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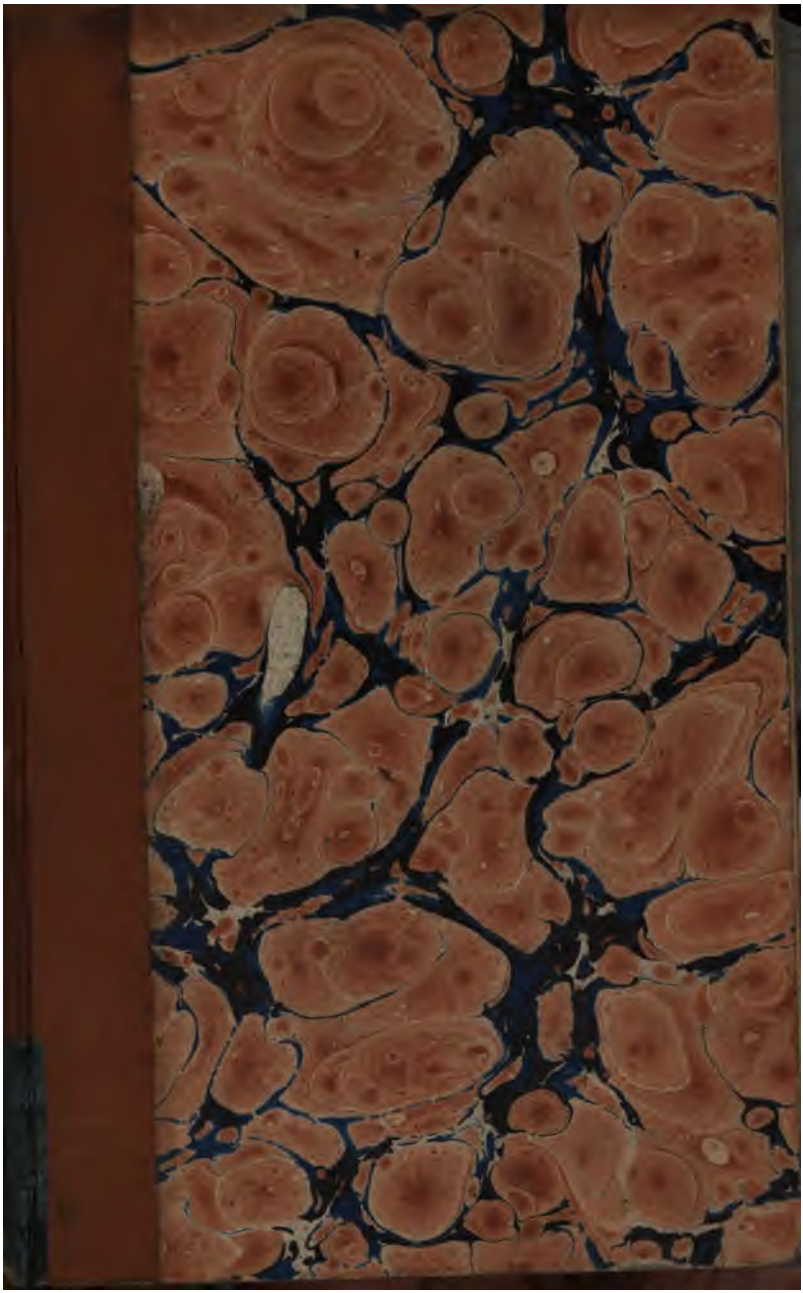

MANUAL

OF

MECHANICS.

BY

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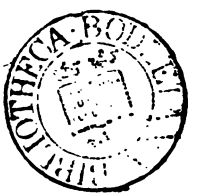

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MECHANICS.

INTRODUCTION.

MECHANICS is the science which treats of the effects of force, whatever may be the source or origin of that force. We are acquainted with two kinds of effects produced by force, viz., pressure and motion. For our knowledge of the first we are indebted to our muscular sense; the latter is made known to us by two of our senses, sight and touch. Mechanics is, therefore, naturally divided into two branches, which treat respectively of these two effects of force. The first branch of Mechanics, which treats of pressure, is called Statics; and the second branch, which treats of motion, is called Dynamics. We shall proceed, without further explanation or definition, to exhibit the elements of these two departments of Mechanics.

STATICS.

CHAPTER I.

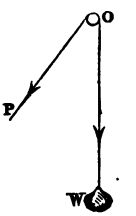
ON THE COMPOSITION OF FORCES.

1. Direction of a Force.—2. Magnitude of a Force.—3. Representation of Forces by Lines.—4. Parallelogram of Forces.—5. Composition of Parallel Forces.—6. Quantity of work done by a Force.

1. **Direction of a force.**—In estimating the effects of a given force, there are two things to be considered:—first, its direction; and secondly, its magnitude. By the direction of a force, we mean the line in which it tends to produce motion, and the force may be conceived to act at any point of this line: as experience fully shows that the effect of a force, whether in producing pressure or motion, is the same at whatever point in the line of its direction it is applied. If, for example, a weight, W, be suspended by a string from the point O, it is found that the same force is required to support it, whether it be suspended at the point C, or B, or A.

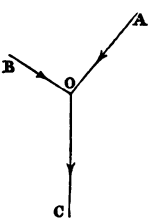
2. **Magnitude of a force.**—The magnitude of a force is estimated numerically by the weight it will counterpoise, and in order to express this we select some unit of weight, such as a grain, a pound avoirdupois, a ton, &c.; and then denote the magnitude of the force by the number of units and decimal parts of a unit it will sustain. If, for example,

a man's hand, acting at P, in the direction OP, sustain a weight W equal to 1 cwt. 2 qrs. 15 lbs., then the force in the direction OP is said to be equal to 183 lbs., or to 1.63392 cwt., according as the unit selected is a pound or a hundred weight. Having obtained its numerical value, we may then compare it with the forces produced by other causes, by first estimating these in a similar manner.



3. **Representation of forces by lines.**—Forces may be represented geometrically by right lines, and this mode of representation has the advantage of exhibiting both direction and magnitude. Having selected a unit of length, such as an inch, a foot, &c., we draw a line in the direction of the given force, and then measure off a segment containing as many units of length as the given force contains units of weight.

Thus, three forces acting at the point O are represented in magnitude and direction by the lines AO, BO, OC, containing, for example, as many feet as there are pounds' weight in the forces represented; while the arrow-heads indicate whether they act from or towards the point O.

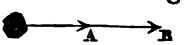


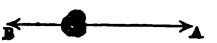
4. **Parallelogram of forces.**—If two forces act upon the same point of a body, it is always possible to find a single force, whose effect shall be the same as the combined effects of the two given forces. This substitution of one force for two others is called the composition of forces.

If the two forces act upon the same point, in the

same line, their combined effect will be equal to the sum or difference of the effects of the forces acting separately, according as they tend to draw the point in the same or in opposite directions.

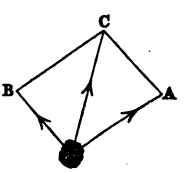
Thus, if OA and OB represent two forces acting upon the point O; in the first case, the effect of the forces will be to draw the body with a force equal to the sum; and in the second case, with a force equal to the difference of the forces OA and OB.





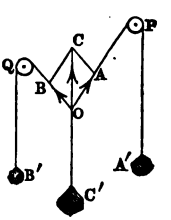
If, however, the forces OA and OB, instead of acting in the same line, act upon the point O in two different directions, then their combined effect is the same as that of a single force OC, whose magnitude and direction is found by the following Proposition:

PROPOSITION I.



If two forces, represented in magnitude and direction by the lines OA *and* OB, *act together upon the same point* O, *their combined effect is the same as the effect of a single force* OC, *which is represented in magnitude and direction by the diagonal of the parallelogram formed by drawing* BC *and* AC *parallel to* OA *and* OB.

Let P and Q be two pulleys (with carefully constructed axles, so as to diminish friction as much as possible), fixed in a vertical wall or board, and having their wheels in the same plane parallel to the wall. Let A′ and B′ be two weights attached to silk cords passing over the



pulleys, and knotted to each other at the point O; at this point let another weight C′ be attached by means of a silk cord. Let the system be now abandoned to its own action, and it will soon settle into a fixed position, represented in the figure.

Forces equal to the weights A′ and B′ now act at the point O, in the directions OP and OQ, and the effect of these is counteracted by the single force equal to the weight C′, acting vertically downwards. Hence it follows, that the effect of the combined forces A′ and B′ is equal to the weight C′ acting vertically upwards. If now the line OA be measured off in the direction OP, containing as many inches as there are ounces in the weight A′; and the line OB be measured off in the direction OQ, containing as many inches as there are ounces in the weight B′, and the parallelogram OACB be completed; then it will be found, on trial, that the line OC is vertical; i. e. it lies in the direction of the force C′, and that it contains as many inches as there are ounces in the weight C′. Hence, if two forces, OA and OB, act upon a point O, their combined effect is equal to the effect of a single force represented in magnitude and direction by the diagonal of the parallelogram OACB.—Q. E. D.

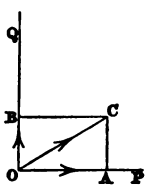
The statement of fact contained in this proposition is usually called the Principle of the Parallelogram of Forces. OA and OB are called Component Forces, and OC is called the Resultant Force.

Care must be taken in constructing the parallelogram of forces, that the components both act *from* the angle of the parallelogram from which the diagonal is drawn.

We are, by means of this law, enabled to replace two forces acting at a point by a single force which

will produce the same effect; this substitution is commonly called the Composition of Forces. It is often necessary, for convenience of proof or illustration, to conceive a single force as divided into two forces acting in different directions; this is called the Resolution of Forces, and is effected by the parallelogram of forces in the following manner :—

Let OC represent the force which we wish to resolve in the directions OP and OQ; from the point C draw the lines CA and CB parallel to OQ and OP; then it is evident from the parallelogram of forces, that OC is the resultant of the forces OA and OB, and that the effect of the forces OA and OB is the same as the effect of the force OC.



For greater simplicity, the angle POQ between the component forces is generally made a right angle.

EXAMPLES.

1. Two equal forces act on the same point, making with each other an angle of 120°. Find the magnitude and direction of the resultant. *Ans.* The resultant bisects the angle, and is equal in magnitude to each of the components.

2. The resultant of two forces acting at right angles is double the lesser of the components. Find the direction of the resultant. *Ans.* It divides the angle between the components into two parts, of which one is 60°, and the other 30°.

3. *If two forces act at right angles, the square of the resultant is equal to the sum of the squares of the components.*

For, if OAC (fig. p. 6) be a right-angled triangle, the square of OC will be equal to the sum of the squares of OA and AC, or OB, which is equal to AC.—Q. E. D.

4. Two forces, equal to 30 and 40 lbs. respectively, act at a point at right angles to each other. Find the resultant.

Ans. 50 lbs.

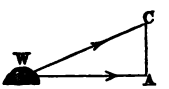
5. Two forces act at a right angle, one equal to 43 lbs., and the other to 22 lbs. Find the resultant. *Ans.* 48.3 lbs.

6. The resultant of two forces, acting at a right angle, is 56 lbs., and one of the components is 31 lbs. Find the other.
Ans. 46.6 lbs.

7. *The effect of a force estimated in a direction different from its own is found by multiplying the force by the cosine of the angle made by it with the direction in which it is to be estimated.*
For (fig. p. 6) the effect of the force OC in the direction OA is the component OA; but OA is equal to OC multiplied by the cosine of AOC.—Q. E. D.

8. Find the effect of a force of 23 lbs. in a direction making with the line of the force an angle of 32°. On referring to the table of sines and cosines, we find cosine 32° = 0.84805; multiplying this decimal into 23 lbs., we obtain for the value of the resolved force 19.5 lbs.

9. A weight W is drawn along the ground by a rope OC, which makes with the horizon an angle COA of 12°. If the force OC required to draw the weight be 142 lbs., a smaller force OA acting horizontally will draw it. Find the horizontal force. *Ans.* 138.89 lbs.

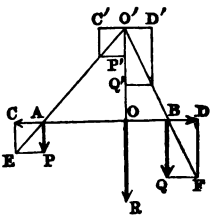


10. Let the force OA be 120 lbs. Find how much this force should be increased, if the weight be drawn at an angle of 17°.
Ans. 5.48 lbs.

11. The resultant of two forces, acting at right angles, divides the angle between them into 37° and 53°, and is equal to 471 lbs. Find the components. *Ans.* 376.15 lbs. and 283.45 lbs.

5. **Composition of parallel forces.**—In section (4) the composition of two forces acting upon the same point of a body has been considered; in this section we shall consider the case of two forces parallel to each other in direction, and applied at two different points of the body. The composition of such forces is contained in the following Proposition :—

PROPOSITION II



If two parallel forces, AP *and* BQ, *act upon the two points* A *and* B *respectively, their combined effect will be equal to the effect of a single force* OR, *equal to their sum, parallel to their direction, and applied at a point* O, *found by dividing the line* AB, *so that* AO *is to* OB *as* BQ *is to* AP, *i. e., inversely as the forces.*

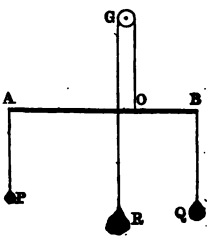
Let two forces, AC and BD, acting in opposite directions, and each equal to the lesser force AP, be introduced into the system. Since these forces would destroy each other if acting alone, they may be introduced without altering the effect of the forces AP and BQ. The forces AP and AC may be compounded (Prop. I.) into a single force AE, and the forces BQ and BD into the force BF; and these resultant forces themselves may be supposed to act at their point of intersection O′. Let them be resolved at this point into their original components, O′P′ and O′C′, O′Q′ and O′D′. The forces O′C′ and O′D′, being equal and opposite, destroy each other, and the resultant is therefore in the direction O′OR, parallel to the forces AP and BQ, and equal to the sum of OP′ and OQ′, or of AP and BQ.

Also, since the triangles BOO′ and BDF are similar, BO is to OO′ as BD is to DF; but OO′ is equal to OA (because the triangles AOO′ and ACE are similar, and AC was made equal to AP or CE); hence BO is to OA as BD is to DF, or as AP is to BQ.—Q. E. D.

EXAMPLES.

The foregoing proposition has been deduced from Proposition 1. by geometrical reasoning. It might have been experimentally demonstrated in the following manner:—

From the extremities of a light strong rod AB, let two weights, P and Q, be suspended, and let a weight R, equal to their sum, be attached to a silk cord, passing over a pulley G; it will be found on trial that the system will remain at rest if the cord be fastened to the point O, selected so that AO shall be to OB in the ratio of the weight Q to the weight P.



1. Two weights of 126 lbs. and 220 lbs. respectively are suspended from the extremities of a straight iron bar, 26 inches in length; find the segments into which the resultant divides the bar. *Ans.* 16.33 inches, and 9.47 inches.

2. Two weights, of 17 stone 6 lbs., and of 2 cwt. 1 qr. 22 lbs., are suspended, as in the last example, to the extremities of a rod 32 feet long; find what point of the rod should be supported in order to sustain the weight.

Ans. The segments are 15.073 feet, and 16.927 feet.

6. **Work done by a force.**—The *work done* by a force is expressed by the weight *lifted* through a given height by the force, and is therefore the product of a weight by a height—

$$\text{Work done} = \text{weight} \times \text{height}.$$

This equation is evidently true, because the work is proportional to the weight raised, and also to the height to which it is raised; if either of these be changed, the work done is changed in the same ratio, and is therefore proportional to the product of the weight and height to which it is raised.

Let W denote the work done, P the weight

raised, and h the height to which it is raised, by the action of a force; then

$$W = Ph. \qquad (1)$$

In this equation P is expressed always in pounds avoirdupois, and h in feet; the unit of work to which W is referred is sometimes called the "foot-pound," to signify one pound lifted through one foot.

If w denote the work done in one minute by an agent, such as a man, horse, &c., working in a particular manner, t the number of minutes the agent is employed, and n the number of agents at work, it is evident that the work done in the time t is

$$W = wnt. \qquad (2)$$

Tables have been constructed, founded on experiment, showing the quantity of work per minute which various agents are capable of accomplishing: from these Tables the value of w is known for each agent, and by their aid many useful questions in practical mechanics may be solved.

If we wish to compare together the work done by different agents, we make use of the following equation:—

$$wnt = w'n't'. \qquad (3)$$

in which the accented letters refer to one set of agents, and the letters without accents to the other.

The following Table is taken from the results of Navier, Morin, and other writers :—

TABLE *of Work done per Minute in Foot-pounds.*

	Duration of Labour.	Work.
MAN.	hours.	ft. lbs.
1. Raising his own weight on a ladder, . . .	8	4222
2. Raising weights with a cord and pulley, . .	6	1559
3. Raising weights with the hand, ...	6	1472
4. Raising weights on the back on a ladder, . .	6	1126
5. Raising weights on a slope with a wheelbarrow,	10	519
6. Shovelling materials to a height of 5 ft. . .	10	467
7. Working inside a treadmill—		
a. On a level with the axis, ...	8	3897
b. Near the bottom of the wheel (24°), . .	8	3637
8. Pushing or drawing horizontally, ...	8	3117
9. Pushing and drawing alternately, vertically, .	10	2380
10. Working a windlass, ...	8	2598
11. Working with the arms and legs, as in rowing,	8	4000
ANIMALS.		
12. Horse yoked to a waggon, walking, ...	10	27279
13. Horse yoked to a carriage, trotting, ...	4.5	41914
14. Horse yoked to a whim, walking, ...	8	17536
15. Horse yoked to a whim, trotting, ...	4.5	25980
16. Ox yoked to a whim, walking, ...	8	15588
17. Mule yoked to a whim, ...	8	11691
18. Ass yoked to a whim, ...	8	4849
19. Steam-horse of Watt and English writers, . .	24	33000
20. Steam-horse of French writers, ...	24	32475

In equation (1) the work done is expressed absolutely as the product of a weight and a height; in equation (2) it is expressed in terms of a number of agents of a given description, acting for a certain time; equating these two values of W, we find

$$Ph = wnt. \qquad (4)$$

In this equation there are five quantities:

P = the *weight* lifted,
h = the *space* through which it is lifted,
w = the *work* done by an agent per minute,
n = the *number* of agents,
t = the *time* during which the agents work;

any four of which being given, the fifth may be found. The following are a few of the numerous important practical questions which may be solved by equations (3) and (4) :—

EXAMPLES.

1. What should be the horse-power of a steam-engine capable of raising 750 tons of coal per day of 12 hours from a pit 100 fathoms deep?

Since the agent here employed is the steam-horse, we have—

$P = 750 \times 2240$ lbs.
$h = 6 \times 100$ ft.
$w = 33000$ ft. lbs.
n = required horse-power.
$t = 60 \times 12$ min.

Ans. 42.4 H. P.

2. An engine of 20 horse-power is employed to pump water to a height of 60 ft. to supply a town of 5000 houses with water, at the rate of 80 gallons per house: how long must the engine work each day?

Here we have, since the gallon of water is 10 lbs.,

$P = 5000 \times 80 \times 10$ lbs.
$h = 60$ ft.
$w = 33000$ ft. lbs.
$n = 20$.
t = required time in minutes.

Ans. $t = 6^{h}.\ 4^{m}$.

3. A horse drawing a waggon at the rate of 2 miles per hour exerts a traction of 154 lbs., what is the work done per minute?

Here we have

$P = 154$ lbs.
$h = 5280 \times 2$ ft.
w = required work.
$n = 1$ horse.
$t = 60$ min.

Ans. $w = 27104$ ft. lbs.

4. Five steam-engines of a total power of 1310 horses were employed in pumping water from the shaft of Dawdon colliery, in Durham; what quantity of water could be raised by them from a depth of 73 fath. per hour?

Here we have

P = required quantity.
$h = 73 \times 6$ ft.
$w = 33000$ ft. lbs.
$n = 1310$.
$t = 60$ m.

Ans. $P = 592191$ galls.

5. Four horses working a whim are required to raise water from a shaft at the rate of 3213 galls. per hour; what is the greatest depth to which the shaft can be unwatered by means of these horses?

Here

$P = 32130$ lbs.
h = required depth.
$w = 17536$ ft. lbs.
$n = 4$.
$t = 60$ m.

Ans. $h = 130.99$ ft.

6. A gang of 20 men is employed to pump water by means of a treadmill to a height of 40 ft.; in what time will they raise 10,000 gallons, supposing one-third of the work applied to be lost by the friction of the pumps?

Here

$P = 10000 \times 10$ lbs.
$h = 40$ ft.
$w = 3897 \times \frac{2}{3}$ ft. lbs.
$n = 20$.
t = required time.

Ans. $t = 1^{h}.\ 17^{m}$.

7. A party of 500 shovellers is employed to throw up an entrenchment in soft ground, and works for 10 hours per day; how many cubic yards of earth will be thrown up in 10 days, supposing each cubic yard to weigh two tons?

P = required weight.
$h = 5$ ft.
$w = 467$ ft. lbs.
$n = 500$.
$t = 6000$ m.

Therefore, &c,

Ans. = 62544 cub. yds.

8. If 30 hodmen be employed on a building 30 feet high to carry bricks to the masons, how many bricks will they raise in 7 hours, allowing 17 bricks to 125 lbs.?

Here

P = required weight of bricks.
h = 30 ft.
w = 1126 ft. lbs.
n = 30.
t = 420 m.

Ans. 64317.

9. Two men are employed to work a windlass, to raise materials from a shaft; they work for 8 hours per day for 10 days; in what time would four men working for 6 hours per day raise the same quantity of materials by ladders from the shaft?

Here we must use equation (3):—

w = 2598 ft. lbs.
n = 2.
t = 10 × 8 × 60 min.
w' = 1126 ft. lbs.
n' = 4.
t' = required time.

Ans. $15^{d}.\ 2^{h}.\ 17^{m}$.

10. A stone quarry is unwatered by the labour of 24 oxen, working 8 at a time or two whims for 8 hours per day; what should be the horse-power of a pumping engine which would do the same work?

Here

w = 15588 ft. lbs.
n = 8.
t = 24 × 60 min.
w' = 33000.
n' = required power of engine.
t' = 24 × 60 min.

Ans. 3.77 H. P.

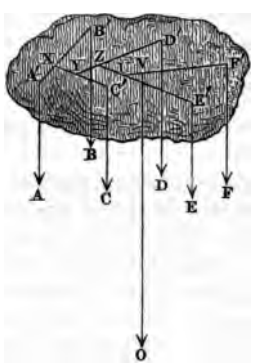
CHAPTER II.

ON THE CENTRE OF GRAVITY.

1. Centre of parallel forces.—2. Centre of gravity.

1. Centre of parallel forces.

To find the resultant of any number of parallel forces acting upon a body.



Let A′A, B′B, C′C, &c. be the given forces. First join A′ and B′, and cut the joining line at X inversely in the ratio of the forces A′A and B′B; the resultant of these forces, which is equal to their sum, passes through this point. (*Vid.* Prop. II.) Next cut the line XC′ in the inverse ratio of the sum of AA′ and BB′ to the force CC′; let Y be the point of section; the resultant of the first three forces AA′, BB′, CC′, which is equal to their sum, passes through this point: next join YD′ and cut it at Z, in the inverse ratio of the sum of AA′, BB′, CC′, to the force DD′; the resultant of the first four forces AA′, BB′, CC′, DD′, which is equal to their sum, passes through this point, and so on. The point V, which is found last, is the point of application of the resultant of all the forces; and the magnitude of the resultant VO is the sum of all the forces AA′, BB′, &c.

*

The point V *which has just been found is called the centre of the system of the parallel forces*, for as it was found without any reference to the direction of the forces AA′, &c., it is independent of this direction, and is consequently a fixed point in the body through which the resultant of the parallel forces must pass, in whatever direction they may tend.

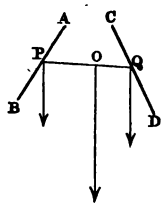
2. **Centre of gravity.**—Gravity is the name given to the force of attraction, exerted by the earth on all bodies situated on its surface: the effects of this force are twofold—statical and dynamical, in virtue of which all bodies have weight, and if unsupported would fall to the ground. The force of gravity, and consequently the weight of bodies, varies from place to place on the surface of the earth; but at each place it is a force exerted upon all bodies and parts of bodies, in lines perpendicular to the horizon, and, consequently, parallel. The direction of the force may be found by means of a plumb-line, or by a sheet of water, since the force of gravity can be proved to be perpendicular to the surface of a fluid at rest.

The weight of a body may be considered as the resultant of the weights of all the atoms or indivisible particles of which it is composed; and since the force of gravity, at the same place on the earth's surface, acts in parallel directions, it follows from what has been proved, respecting a system of parallel forces, that the resultant of the weights of all the parts of a body passes through a fixed point, which is the centre of this system of parallel forces. This point is called the *centre of gravity*. The method of finding the centre of a system of parallel forces having been described in the pre-

ceding paragraph, it is not necessary here to repeat it, as it is plain that the problem of finding the centre of gravity of a body is the same as that of finding the centre of a system of parallel forces.

EXAMPLES.

1. To find the common centre of gravity of two homogeneous lines. Let AB and CD be the given lines; since they are homogeneous, the centre of gravity of each line will be its point of bisection P and Q; join these two points, and cut the joining line inversely as the weights of AB and CD, i. e. so that OP will be to OQ as the weight of CD is to the weight of AB. The truth of this construction is evident, from the consideration that there are two forces parallel to each other, and equal to the weights of AB and CD, applied at the points P and Q, and by Prop. II. the point O will be the point of application of their resultant.

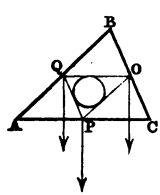


2. Let the weights of AB and CD, respectively, be 23 lbs. and 42 lbs., and the length of PQ 14 inches. Find the line OQ to three places of decimals. *Ans.* 4.953 in.

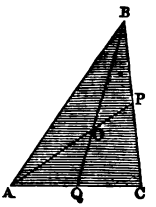
3. Let the weights be 3 cwts. 2 qrs. 15 lbs., and 1 cwt. 3 qrs. 25 lbs., and the line PQ equal 3 feet, 7 inches. Calculate the line OP to three places of decimals of feet. *Ans.* 1.261 ft.

4. To find the common centre of gravity of three bodies. Join the centres of gravity of the first and second bodies, and cut the joining line inversely as their weights. Join this point with the centre of gravity of the third body, and cut the joining line inversely in the ratio of the sum of the weights of the first and second bodies to the weight of the third. The point thus found is the centre of gravity required.

5. To find the centre of gravity of the perimeter of a triangle. Let ABC be the given triangle. Bisect each of the sides in O, P, Q; the weights of the three sides respectively may be conceived as concentrated in these three points. The question is thus reduced to Example (4). It is not difficult to prove that the centre of gravity is the centre of the circle inscribed in the triangle OPQ.



6. To find the centre of gravity of a homogeneous thin plate, cut into a form of a triangle. Let ABC be the triangle, which may be conceived as divided into thin bars parallel to AC. The line BQ, joining the angle B with the point of bisection Q of the opposite side, will pass through the centres of gravity, or points of bisection of all these bars. Consequently, the centre of gravity of the triangle lies on this line BQ. In a similar manner it may be shown to lie on the line AP, which is drawn bisecting the side BC. Consequently, it must be at the intersection O of these lines.

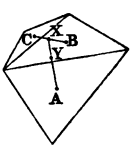


7. *If a body be suspended from a fixed point* O, *it will not be in equilibrium unless the line* OG, *joining the point of suspension with the centre of gravity, be vertical.*

For, let the line OG not be vertical, and let GC represent the weight of the body acting vertically downwards; let this force be resolved into two forces, GA and GB (Prop. I.); it is evident that the force GA will be destroyed by the reaction of the fixed point O, while the force GB will cause the body to move towards the vertical line OV. Hence the body cannot be at rest unless the line GO be vertical.

8. Hence may be deduced an easy mode of finding, experimentally, the centre of gravity of a plain figure of irregular shape. Let it be suspended to a fixed point, by means of a cord attached to any point of its surface; when it has arrived at a state of rest the line of the cord will be vertical, and will also pass through the centre of gravity. By marking on the body the line of the cord, and producing it downwards, we shall obtain one line passing through the centre of gravity. Repeating the experiment, with a different point of suspension, we shall obtain a second line containing the centre of gravity; consequently, the intersection of these two lines will be the centre of gravity of the body.

9. To find the centre of gravity of a polygon. Let the polygon be divided into triangles by diagonals drawn from any angle, and let the centre of gravity of each of these triangles be found (Ex. 6); let A, B, C be the centres of gravity; join BC, and cut the joining line in X inversely as the weights of the first two triangles; join XA, and cut it in Y inversely as the sum of the weights of the first two triangles to the weight of the third triangle. The point Y is the centre of gravity of the polygon.



CHAPTER III.

ON MACHINES IN EQUILIBRIUM.

1. Principle on which Equilibrium of Machines is determined.—2. The Lever.—3. The Wheel and Axle.—4. The Inclined Plane.—5. The Moveable Inclined Plane.—6. The Screw.—7. The Pulley.

1. **Principle on which Equilibrium of machines is determined.**—A machine is an instrument by means of which pressure or motion may be transmitted from one point to another, and altered both in magnitude and direction. In all machines there are two forces to be considered: first, the Power applied to the machine; and secondly, the Resistance, to overcome which the Power is used. Two forces will not equilibrate each other unless they be equal and opposite, from which it might be supposed that the Power and Resistance should be equal to each other. This would necessarily be the case if the machine were free, i. e. not in contact with fixed points or surfaces. But as all machines contain fixed points or fulcra, and sometimes fixed lines or surfaces, and as these are capable of resisting pressure, and so supporting part of the Resistance, it follows that the Power may be less than the Resistance, and bear any ratio to it, according to the position of the fixed points or surfaces. It is to be observed, that while fixed points are capable of resisting pressure in all directions, fixed lines and surfaces can only do so in a direction at right angles to themselves. The principle, according to which the equilibrium of all machines is to be found, is contained in the following Proposition:—

PROPOSITION III.

Let the Power and Resistance be considered as two forces applied to the machine, and let the direction of their resultant be determined; the machine will be in equilibrium when this resultant passes through a fixed point, or is perpendicular to a fixed line or surface in the system.

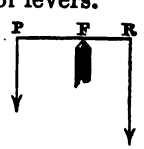
The truth of this Proposition is evident from the consideration, that if the resultant do not pass through a fixed point, &c., it will cause motion, and consequently the machine cannot be in equilibrium.

COROLLARY.—The strain or pressure sustained by the fixed point or surface is the resultant of the Power and Resistance.

The first machine to be considered is the lever.

2. **The Lever.**—A lever is a rigid bar, capable of motion about a fixed point or axis, which is called the fulcrum. There are three kinds of levers.

In the first kind, the fulcrum F is situated between the Points P and R, at which the Power and Resistance are applied.



In the second kind, the Resistance is applied at a point situated between the fulcrum and the point at which the Power is applied.

In the third kind, the point at which the power is applied is situated between the fulcrum and the point at which the resistance is applied.

In each of these cases there is a fixed point (viz. the fulcrum F) in the system; consequently, by Prop. III. the resultant of the Power and Resistance must pass through this point, which therefore supports a strain whose magnitude is measured by this resultant.

If the Power and Resistance be parallel in direction, and be applied perpendicularly to a straight lever, as in fig. p. 20,

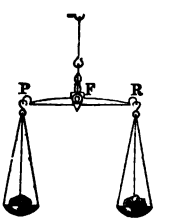
The Power is to the Resistance inversely as the arms of the lever.

For, by Prop. III. the resultant of the forces P and R passes through the point F, and therefore by Prop. II.

$$P : R :: \mathrm{FR} : \mathrm{FP}.$$

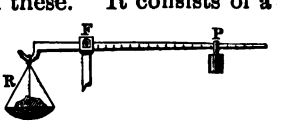
—Q. E. D.

If the arms of the lever be equal, then it is requisite for equilibrium that the Power and Resistance shall be also equal.

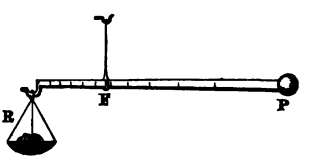


This case occurs in the common balance, in which the Power and Resistance are the weights in each pan of the scales; if the balance be true, the arms are equal, and therefore the weights are also equal; but if the arms be unequal, the weights will be also unequal in the inverse ratio. The simplest mode of detecting a false balance is to transpose the weights; for they cannot counterbalance each other, when so transposed, unless the arms and weights be both equal.

Besides the balance with equal arms, there are other balances in common use for ascertaining the weights of bodies for commercial purposes. The *Roman Steelyard* is one of these. It consists of a beam, or bar of iron, resting upon a pivot F, and having one arm longer than the other. To the extremity of the shorter arm is attached a scale-pan, and the position of the pivot is so adjusted that the weight of the scale-pan and shorter arm balances



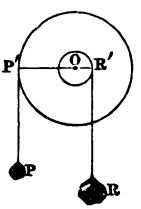
the weight of the longer arm; a weight or counterpoise P can be moved along the longer arm of the balance, while the commodity to be weighed is placed in the scale-pan. When the counterpoise balances the contents of the scale-pan, it is evident, from the principle of the lever, that their weights will be as RF to FP; but the first and third terms of this proportion are constant, viz., the counterpoise P, and the distance RF; therefore, the second and fourth terms of the proportion bear a constant ratio to each other; i. e. the commodity to be weighed is proportional to the line FP; by dividing the latter into equal parts, we can therefore determine the weight of a given article.



Another balance of this kind in common use is the *Danish Balance*, which differs from the *Roman Steelyard* in having its fulcrum F moveable, and the counterpoise P fixed at one extremity; to the other extremity is attached a scale-pan. In order to graduate the beam, we place successively 1, 2, 3, &c. lbs. in the scale-pan, and mark the point on the beam at which the fulcrum must be held so that the beam shall be horizontal; at these points we engrave the figures 1, 2, 3, &c.; the intervals between these figures must be divided by adding ounces to the corresponding pounds weight in the scale-pan, and so dividing the intervals experimentally. The intervals between the figures 1, 2, 3, &c., will not be equal, as in the case of the *Roman Steelyard*.

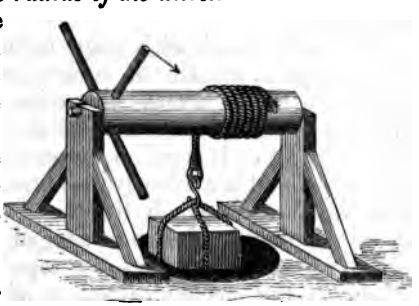
3. **The wheel and axle.**—The wheel and axle is one of the simplest and most useful of the modifications of the lever; it consists, as its name expresses,

of a large wheel, or simply spokes fastened perpendicularly on an axle, which is supported by two rings or sockets, in which it revolves, which act as fulcra, and must sustain the pressure caused by the resultant of the Power and Resistance. The Power is applied to the circumference of the wheel; and the Resistance to the circumference of the axle; in directions at right angles to their radii.



The annexed figure represents a cross section of the wheel and axle, in which OP′ is the radius of the wheel, OR′ the radius of the axle, and P and R the Power and Resistance, both acting vertically downwards. It is evident, from inspection of the figure, that P′R′ may be considered as a lever of the first order, having its fulcrum at O, and the weights P and R applied at its extremities; hence, the condition requisite for equilibrium in the wheel and axle may be expressed as follows:—

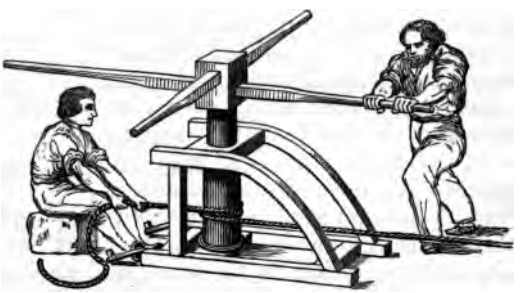
There will be Equilibrium in the wheel and axle, when the Power is to the Resistance as the radius of the axle is to the radius of the wheel.



One of the most common examples of the wheel and axle is the *windlass*, in which the Power is applied by means of a winch handle or spokes. If the labourer who turns the handle exert his force always at right

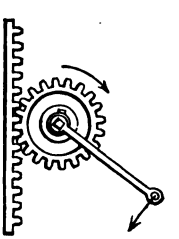
angles to it, the conditions of equilibrium in the windlass will be precisely the same as in the wheel and axle; for the winch handle describes a circle which corresponds with the wheel, provided the force act perpendicularly to the radius.

If the axis be vertical, and the wheel consist of moveable spokes against which a pressure is exerted by means of men pushing with their shoulders, the



instrument is called a *capstan*, and the conditions of equilibrium are the same as in the wheel and axle. This instrument is used on board ship for raising the anchor.

If a small wheel with cogs, or teeth, be placed upon the axle, and this be made to work into a vertical bar fitted with cogs, this combination is also an instance of the wheel and axle; it is called the *rack and pinion*, and is evidently the same as a windlass, in which the Resistance is applied by means of cogged wheels instead of by means of a rope or chain; the mechanical effect of both instruments is the same.



EXAMPLES.

1. What force will be required to work the handle of a windlass, the Resistance to be overcome being 1156 lbs., the radius of the axle being 6 inches, and of the handle 2 feet 8 inches?

Ans. 216.75 lbs.

2. A weight of 17 lbs. just balances a weight of 79 lbs. on a wheel and axle. What will be the radius of the axle, if that of the wheel be 17 inches? *Ans.* 3.65 inches.

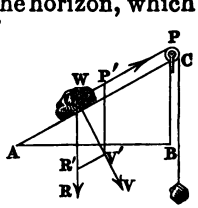
3. Sixteen sailors, exerting each a force of 29 lbs., push a capstan with a length of lever equal to 8 feet, the radius of the capstan being 1 ft. 2 inches. Calculate the Resistance which this force is capable of sustaining. *Ans.* 1 ton, 8 cwt. 1 qr. 17 lbs.

4. Supposing them to have wound the rope round the capstan, so as that it doubles back on itself, the radius of the axle is thus increased by the thickness of the rope. If this be 2 inches, how much will the Power of the instrument be diminished?

Ans. By one-eighth; or $12\frac{1}{2}$ per cent.

5. The Resistance of a sluice-gate, to which a cogged rack is attached, amounts to 1 ton; if this be raised by means of a winch and pinion, having for radii 1 ft. 7 inches, and 2.5 inches, respectively, calculate the power which must be applied to the handle, so as just to sustain the Resistance. *Ans.* 294.73 lbs.

4. **The inclined plane.**—The inclined plane is a plane which makes an angle with the horizon, which angle is called the inclination of the plane; thus if AC be an inclined plane, and AB be a horizontal plane, the angle BAC is called the inclination of the plane AC; and if from any point C a perpendicular CB be let fall upon the horizontal plane, the line CB is called the height of the plane, AB is called the base, and AC the length of the plane. It is usual with engineers to measure the inclination of a plane by the ratio which its height bears to its length; i.e. by the sine of the inclination. For example, if CB be the tenth part



of AC, the inclination of the plane is expressed by saying that it is one in ten; i.e. a man should walk ten yards up the plane before he would have ascended one yard in vertical height; the inclination of a plane, when so expressed, is sometimes called the gradient. In the inclined plane we have an example of a fixed surface in a machine; this surface will destroy all pressure applied to it in a perpendicular direction, and by means of this consideration we can easily solve the problem of the inclined plane.

This problem may be thus stated: a heavy body being placed upon an inclined plane; given the direction of the force applied to it, to find the magnitude of the Power which will sustain it on the plane.

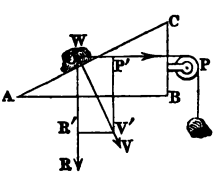
The general case of this problem will be discussed in the examples appended to this section; for general readers it will be sufficient to discuss the two cases which most commonly occur: viz.—the case in which the Power is applied in a direction parallel to the plane, and the case in which the direction of the Power is parallel to the base of the plane.

1st Case.—Let W. (fig. p. 25) be a body placed on an inclined plane AC; let a force be applied to sustain it, by means of a cord passing over a pulley P, in the direction OP, parallel to the plane; and for simplicity let us suppose that the cord is attached to the centre of gravity O, of the body W.

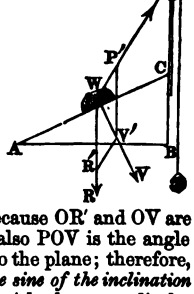
The Power in this case acts in the direction OP, and the Resistance or weight in the vertical direction OR; by Prop. III. the resultant of these two forces must be perpendicular to the plane AC; let OV be drawn perpendicular to the plane. The

problem is therefore reduced to the following: knowing the directions OP and OR of two forces, and the direction OV of their resultant; also knowing one of the forces in magnitude, viz., the weight W; to calculate the magnitude of the other force and resultant. This may be done as follows: let OR′ be measured off, containing as many inches as there are pounds in W, and let the parallelogram R′OP′V′ be completed; then OP′ will represent the Power, and OV′ the resultant, which, by Coroll. Prop. III. is equal to the pressure on the plane; but the triangle OP′V′ is similar to ACB, because P′OV′ is right, and OP′V′ equal to ACB, the lines P′V′ and CB being parallel; therefore OP′ is to P′V′ or OR′, as BC is to AC; or *the Power is to the Resistance as the height of the plane to its length.*

2nd Case.—Let W be a body supported on an inclined plane AC by means of a force acting in the direction OP, parallel to the base of the plane. By what was stated in the preceding case, it appears that the Power and Resistance, and their resultant, act in the known directions OP, OR, and OV; and that the Resistance, or weight W, is known in magnitude; let OR′ be measured off as before, to represent it, and the parallelogram OR′V′P′ completed; then OP′ and OV′ will represent the Power and resultant pressure on the plane AC; but the triangle OP′V′ is similar to ABC: therefore OP′ is to P′V′ or OR′, as CB is to BA; i. e. *The Power is to the Resistance, as the height of the plane to its base.*



EXAMPLES.



Let the force be applied in any direction OP, to sustain the body W on the inclined plane; then, as before, the directions of the Power, Resistance, and resultant are known, viz., OP, OR, and OV; and the magnitude of the Resistance is known; therefore, the magnitude of the Power and resultant can be calculated: let OR′ represent the Resistance, then completing the parallelogram OR′V′P′, OP′ and OV′ will represent the Power and pressure on the plane; but OP′ is to P′V′ or OR′, as sine ROV is to sine POV; and ROV is equal to the inclination CAB, because OR′ and OV are perpendicular to AB and AC respectively; also POV is the angle made by the Power with the perpendicular to the plane; therefore, finally,—*The Power is to the Resistance, as the sine of the inclination is to the sine of the angle made by the Power with the perpendicular to the plane.*

1. A road has a rise of 1 yard in 62; calculate the number of minutes and seconds in its inclination. *Ans.* 55′ 27″.

2. If the inclination of a road be 4°, find a number which shall bear to unity the ratio of the length to the height. *Ans.* 14·33.

3. The longest inclined plane in the world is the road from Lima to Callao, in S. America; it is 6 miles long, and the fall is 511 feet. Calculate the inclination. *Ans.* 55′ 27″, or 1 yard in 62.

4. If the force required to draw a waggon on a horizontal road be $\frac{1}{21}$st part of the weight of the waggon; what will be the force required to draw it up a hill, the slope of which is 1 in 43?
Ans. $\frac{1}{14 \cdot 11}$th part of the weight.

5. If the force required to draw a train of carriages on a level railroad be $\frac{1}{200}$th part of the load; find the force required to ascend a gradient of 1 in 56. *Ans.* $\frac{1}{43 \cdot 75}$th part of load.

6. What force is required (neglecting friction) to roll a cask weighing 964 lbs. into a cart 3 feet high, by means of a plank 14 feet long resting against the cart?
Ans. the force must exceed 206 lbs.

7. Let the angle POV be 101°, and the inclination of the plane 6°; if the weight W be 2 cwt. 1 qr. 17 lbs.—find the Power necessary to sustain it. *Ans.* 28.64 lbs.

8. If the angle POV be 120°, the inclination 2°, and the weight 31 tons; find the power. *Ans.* 1 ton, 4 cwt. 3 qrs. 26 lbs.

9. Let the angle POV be 167°, and the inclination 1 in 26, the weight being 22 tons; find Power. *Ans.* 3.761 tons.

10. If a waggon weighing 2 tons, 14 cwt. rest upon an incline of 1 in 27; find the pressure upon the road.

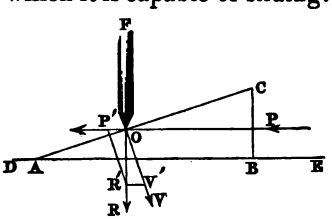
Ans. 2 tons, 13 cwt. 3 qrs. 23.8 lbs.

11. If the force required to overcome friction at a given speed upon a horizontal railroad be 10 lbs. per ton; and the similar force upon a turnpike road 236 lbs. per ton, find the effect produced upon both by a gradient of 1 in 21, and show why the gradients on railroads should be much less than on common roads.

Ans. The force on railroad is increased to 116.6 lbs. per ton.
——— turnpike ,, 342.6 lbs. per ton.

Hence, it appears that the force on the railroad should be increased more than eleven times, while the force on the turnpike road should be increased less than one and a half times.

5. **The moveable inclined plane.**—The inclined plane, considered in Section 4, was a fixed plane of indefinite dimensions: we shall now consider an inclined plane of finite dimensions, placed upon a horizontal plane, upon which it is capable of sliding. Let DE be the horizontal plane on which the moveable inclined plane ABC is placed. Let F be a rod through which the Resistance is applied to the plane at the point O, and let us suppose, for simplicity, that the Power is applied to the back of the plane in the direction PO, parallel to the base AB. In this system we have two surfaces, DE and AC, one fixed and the other moveable, but each capable of sustaining pressure in a perpendicular direction only. The Resistance acting through the rod F must therefore produce its effect

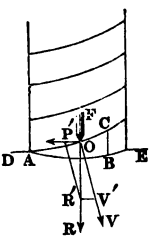


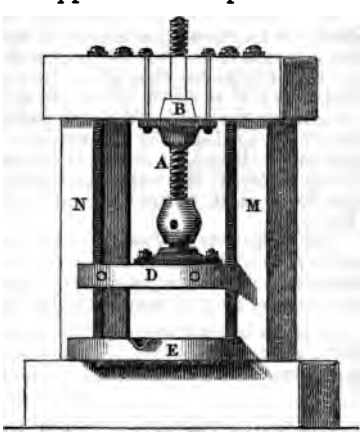
upon the plane in the direction OV; while the pressure resulting from the Power and Resistance must be sustained by the fixed plane DE, and act in the direction OR. We know, therefore, the directions of the Power, and of the two pressures on the planes AC and DE; and we also know the magnitude of the Power; we can therefore calculate the magnitude of the pressures on AC and DE as follows: Measure off the line OP′, containing as many inches as there are pounds pressure in the Power, and complete the parallelogram P′OR′V′; OV′ and OR′ will represent in magnitude the pressures on the planes AC and DE. But the triangle P′OR′ is similar to ABC, and therefore, OP′ is to OR′ as BC is to BA; i.e. *The Power is to the Pressure on the horizontal plane as the height of the moveable plane is to its base.* In like manner it appears that OP′ is to OV′, as BC is to CA; i.e. *The Power is to the Pressure on the moveable plane as the height of the plane to its length.*

The moveable inclined plane, which has just been described, is the wedge, provided the Power applied be a pressure, and not a blow; but as the latter is the practical method of using a wedge, and as the theory of the moveable inclined plane does not apply to such forces as impacts, we shall omit the consideration of the wedge, and proceed to a most important example of the moveable inclined plane, which is called—

6. **The Screw.**—The screw is a moveable inclined plane wrapped round a circular cylinder. In order to explain the mode of its action, we shall suppose the figure in p. 29 to be wrapped round a vertical cylinder. The base DE being fixed, if a Power OP′ tending to turn the screw round be applied, the Resistance will react at the point O in two directions

with pressures measured by OV′ and OR′; the latter being the pressure sustained by the horizontal plane; and the former, the pressure perpendicular to the thread of the screw; hence, as before, OP′ is to OR′ as CB is to BA; but when the plane is wrapped completely round the cylinder, the interval between two threads will be to the circumference of the cylinder as CB is to BA. Hence, *The Power applied at the circumference of cylinder is to the vertical Resistance, as the interval between the threads is to the circumference of the cylinder.* Take for example the screw-press; resistances similar to F are applied at each point of the thread of the screw A, by means of a nut B, which is tapped with a screw that fits into the thread cut on the cylinder; this nut, together with the uprights M and N, supports the resistance caused by the substance pressed between the plates D and E, and the power is applied at the extremity of a lever attached to the screw. This power bears to the power at the circumference of the screw the same ratio that the radius of the screw bears to the radius of the circle described by the Power,





on the principle of the wheel and axle, p. 23; or the same ratio that the circumference of the screw bears to the circumference of the circle described by the Power. Hence, the Power is to the Resistance parallel to the axis of the screw, in a ratio compounded of the ratio of the interval between the threads to the circumference of the cylinder, and of the circumference of cylinder to the circumference of circle described by Power; i. e. *The Power is to the Resistance parallel to the axis, as the interval between the threads is to the circumference of circle described by the Power.*

EXAMPLES.

1. *To find the ratio which the Power bears to the Pressure perpendicular to the thread of a Screw.*—It appears from section 5, that the Power is to the Pressure, on the moveable inclined plane, as the height to the length of plane; hence, in the screw (fig. p. 31), PO is to OV′ as CB is to CA; or as the interval between the threads is to the length of thread in one turn of the screw. Hence, the Power applied at the circumference of cylinder is to the pressure on the thread, as the interval between the threads is to the length of thread; this ratio must be compounded with the ratio of the Power at O, to the Power at the circumference of cylinder. Hence,—

The Power is to the Pressure on the thread, in a ratio compounded of the ratio of the interval between the threads to the length of thread in one turn of the screw; and of the ratio of the radius of the screw to the radius of the circle described by the Power.

2. If the interval between the threads of a screw be $\frac{1}{11}$th of an inch, and the circumference of the circle described by the Power be 2 feet; what Resistance will a Power of 13 lbs. sustain?

Ans. 1 ton, 10 cwt. 2 qrs. 16 lbs.

2. If the interval be $\frac{1}{8}$th of an inch, and the diameter of the circle described by the Power be 2 feet; find what Resistance a Power of 17 lbs. will sustain. *Ans.* 5 tons, 3 cwt.

4. If the interval be $\frac{2}{3}$ths of an inch, and the radius of the circle described by the Power be 2 feet; find Resistance which a Power of 32 cwt will sustain. *Ans.* 643 tons, 8 cwt.

5. If there be 19 turns of a screw in $1\frac{4}{5}$th inch, and if the handle, 1 ft. 5 in. in length, be worked with a Power of 2 cwt. 1 qr. 17 lbs.; find the Resistance. *Ans.* 149.9 ton.

6. The interval between the threads is $\frac{1}{8}$th of an inch, and the diameter of the cylinder is one inch, and a Power equal to 139 lbs. acts in a circle whose circumference is 3 feet; find the Pressure on the thread of the screw. The Pressure is, by Example 1, equal to the Power, multiplied by the ratio of length of the thread to the interval between the threads, and of the circumference of the circle described by Power to the circumference of the cylinder; i. e. equal to 139 lbs. multiplied by $\frac{3.1460}{0.1666}$ (the numerator of this fraction is the length of the thread, which is equal to the square root of the sum of the squares of the interval between the threads and of the circumference of the cylinder), multiplied by $\frac{36}{3.14159}$, equal to *Ans.* 30066 lbs.

7. The interval between the threads of a screw being $\frac{1}{11}$th of an inch, and the diam. of cylinder $\frac{3}{5}$th of inch; find length of thread in 14 revolutions. *Ans.* 16.512 inches.

8. The interval being $\frac{3}{5}$th inch, and the diam. $\frac{5}{8}$ inch; find length of thread in one revolution. *Ans.* 5.90 in.

9. If the Power be applied in Example 7, with a radius of 3 inches, and equal to 14 lbs.; find pressure on thread.
Ans. 1 ton, 5 cwt. 3 qrs. 22 lbs.

10. In Example 8, let the Power be 37 lbs. applied, with a radius of 1 ft. 2 inches; find Pressure on thread.
Ans. 3 tons, 17 cwt. 2 qrs. 13 lbs.

11. Find, in Examples 9 and 10, the Resistance parallel to the axis of the screw. *Ans.* 5 cwt. 3 qrs. 18 lbs.
—— 3 tons, 17 cwt. 1 qr. 27 lbs.

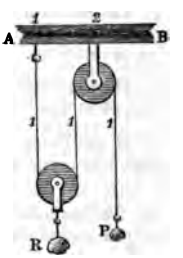
12. A pressure of 5 tons is exerted at the back of a moveable inclined plane, having an inclination of 2°; find the weight which this force will sustain. *Ans.* 143 tons, 3 cwt. 2 qrs. 21 lbs.

13. Find, in this case, the pressure sustained by the inclined plane. *Ans.* 143 tons, 5 cwt. 3 qrs. 8 lbs.

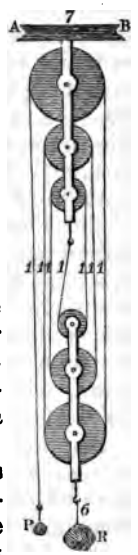
7. **The Pulley.**—A force may be transmitted from one point to another by means of a cord, and if there were no friction, this force might be transmitted undiminished in any direction. In

order to diminish friction in thus transmitting a force from one point to another, by means of a cord, we make the cord pass over a wheel with a groove cut in its circumference, which turns with the cord, and so diminishes the friction. This wheel is called the *sheaf*, and the wooden frame in which it works, and to which its axle or pivot is fastened, is called the *block* of the pulley. The principle on which the equilibrium of a system of pulleys is determined is the following :—

The tension of the cord passing through the system is constant, and equal to the Power applied at its extremity.

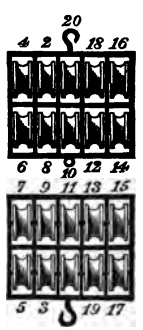


The *single moveable Pulley* is represented in the annexed figure, which is intelligible without any description. The power applied at P produces a constant tension through the whole cord; this is denoted by the figure 1 placed beside each portion of the cord. The weight R is attached to the moveable pulley, and since the block of this pulley is supported by two tensions, parallel in direction, and each equal to the Power, it is plain that this block is capable of sustaining a weight equal to twice the Power applied. The fixed beam AB must support, by Prop. III., a strain equal to the resultant of the Power and Resistance; and since these are parallel in direction, the strain on the beam will be three times the Power. This is shown by the figures annexed to the points at which the cord and fixed pulley are attached, which express the strain on each of these points.

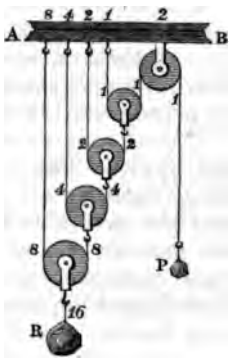


The single moveable pulley may be extended by placing in each block several sheaves, and passing the cord over the upper sheaf in the fixed block, then under the lower sheaf in the moveable block, and so on, as represented in the figure. The tension of the string being constant and equal to the Power, and the moveable block supported by six tensions, we shall have the Resistance sustained equal to six times the Power; and in general, where there are two blocks, one fixed and the other moveable, *the Resistance or weight supported will be equal to the Power multiplied by twice the number of sheaves in the block.*

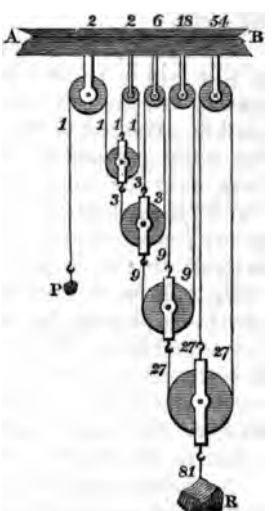
The pulley invented by Smeaton is of this kind; its two blocks are represented in the annexed figure. The cord is attached to the upper block at the point marked 10, it passes under the sheaf 1, then over the sheaf 2, 3, &c. and the Power is applied at its extremity after it has passed over the sheaf 20. It is evident that in this system of pulleys there will be twenty cords sustaining the lower block, having each a tension equal to the Power applied. The Resistance is therefore equal to twenty times the Power. The fixed beam will in this case sustain a pressure equal to twenty-one times the Power.



A system of pulleys which contains more than one cord is called a *Burton.*



One kind of Burton is represented in the annexed figure, in which there are four cords, sustaining tensions represented by 1, 2, 4, and 8 times the Power respectively; the weight sustained being equal to 16 times the Power. As the sum of the Power and Resistance is therefore 17 times the Power, this must be sustained by the beam AB; the manner in which this is done is represented by the numbers placed over the beam, which express the strain produced at each point of attachment.



Another kind of Burton is represented in the annexed figure, in which also there are four cords; but each cord, instead of being fastened to the beam AB, is made to pass over a fixed pulley, and is then attached to the upper part of its own block. The tensions supported by these cords are 1, 3, 9, and 27 times the Power respectively. The weight supported is 81 times the Power. The numbers above the beam AB denote the strain on each point of support; their sum is equal to 82, i. e. equal to the sum of the Power and Resistance.

EXAMPLES.

1. What Power will be required in a Burton of the first kind with 4 moveable pulleys to sustain a weight of 17 tons, 12 cwt., neglecting friction, the rigidity of the cord, and the weight of the blocks. *Ans.* 1 ton, 2 cwt.

2. What Power would be required in a Burton of the second kind with 4 moveable pulleys to sustain the same weight?
Ans. 4 cwt. 1 qr. 10 lbs.

3. If a Burton of the first kind contain 11 pulleys, what weight will 13 lbs. support by means of it?
Ans. 11 tons, 17 cwt. 2 qrs. 24 lbs.

4. A Burton of the second kind contains 7 pulleys, what weight will 17 lbs. support by means of it?
Ans. 16 tons, 11 cwt. 3 qrs. 23 lbs.

5. Compare the efficiency of Smeaton's block with a Burton of the second kind with 5 moveable pulleys. *Ans.* 12.15 to 1.

6. Compare the efficiency of the two kinds of Burtons, each having 10 moveable pulleys. *Ans.* 57.6 to 1.

The machines which have been described in this chapter are among the simplest and most useful of those which occur in practice. There are many combinations of these machines with each other and with other machines, which could not be described without entering into details unsuited to the limits we have prescribed to ourselves in this work. Those which have been described are sufficient to illustrate Prop. III., and to show the method to be followed in investigating the relation between the Power and Resistance in other cases.

CHAPTER IV.

ON MACHINES MOVING UNIFORMLY.

1. Constancy of Work done by a Force.—2. The Lever.—3. The Wheel and Axle.—4. The Inclined Plane.—5. The Screw.—6. The Pulley.—7. The Gain in Power is Loss in Time.

In the preceding chapter we have considered machines in a state of equilibrium, but as they are generally employed in a state of uniform motion, it is necessary to consider them in this point of view. When a machine passes from a state of rest into a state of uniform motion, the power and resistance each move through a certain space in a given time, and the *work done* by them is measured by their pressure in pounds multiplied into the space in feet through which they move in one minute (p. 9). The principle according to which the equilibrium of machines in a state of uniform motion is determined is called the principle of the constancy of work done, and is explained in the following paragraph.

1. **Constancy of work done by a force.**—A force may be transmitted from one point of application to another by means of various instruments, such as cords, bars, wheels, &c.; but it is not possible for these inanimate objects to generate work, and whatever diminution the work done by the force may undergo by friction, we are certain that it can never be increased. We always find, in fact, that the *work applied* to any machine is greater than the *work done* by the machine, and are led by nu-

merous experiments and observations to conclude that this loss of work is due to friction; so that if it were possible to construct a machine without friction, the work applied would be equal to the work done, and the celebrated problem of "perpetual motion" would be mechanically possible. If we imagine a machine moving uniformly without friction, the work done by any part of the machine is constant and equal to the work applied to move the machine. This is called the "principle of constancy of work done," and is applicable to machines moving uniformly without friction.

The principle of constancy of work may be thus stated:—

Let P be the power acting at any part of a machine;

Let p be the distance through which it moves in one minute in its own direction;

Let R be the resistance acting at some other part of the machine;

Let r be the distance through which it moves in one minute; then

$$Pp = Rr. \qquad (5)$$

In this equation Pp and Rr are the work done by the power and resistance, and they are equal to each other when the machine moves uniformly without friction.

2. **The Lever.**—If a lever (p. 20) turn uniformly round its fulcrum F, at the end of any time, the power and resistance will have passed through arcs of circles, whose lengths are proportional to their radii FP and FR. Substituting these arcs or the radii FP and FR for p and r in equation (5), we find

$$\text{Power} \times \text{PF} = \text{Resistance} \times \text{RF};$$

or Power : Resistance :: RF : PF ;

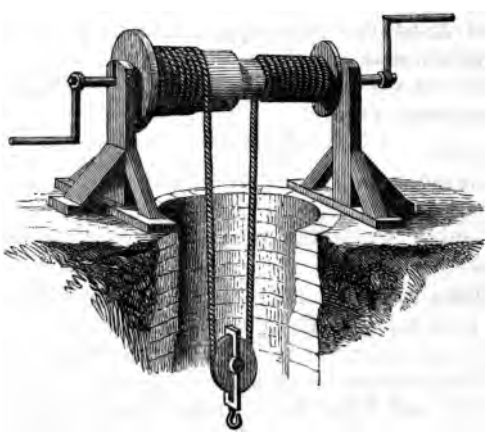
i. e. *the power is to the resistance inversely as the arms of the lever.*—Q. E. D.

3. **The Wheel and Axle.**—If the wheel and axle (p. 23) revolve uniformly, in a single revolution, the power passes through a space p, which is the circumference of the wheel, and the resistance, in the same time, passes through a space r, which is the circumference of the axle. Substituting these values for p and r in equation (5), we find,

Power × circumference of wheel = Resistance × circumference of axle;

Or, since the circumferences of circles are proportional to their radii,

The power is to the resistance as the radius of the axle is to the radius of the wheel —Q. E. D.



The wheel and axle is sometimes modified, as shown in the accompanying figure. In this case

the work done by the power in the time of a single revolution is equal to the weight lifted multiplied by the difference between the circumferences of the two axles, because the weight in that time ascends through a space equal to the circumference of the larger axle, and descends through a space equal to the circumference of the smaller. Hence,

The power is to the weight lifted as the difference of the circumferences of the axles is to the circumference of the winch handle.

4. **The Inclined Plane.**—The only case of the inclined plane which is of any practical importance is that in which the power is applied parallel to the plane, as in the case of drawing loads up hills, &c. (p. 25). In this case, while the power moves through the length of the plane, the weight is lifted through the height of the plane, and therefore, by equation (5),

Power × length of plane = Resistance × height of plane.

Or,

The power is to the resistance as the height of the plane is to its length.—Q. E. D.

5. **The Screw.**—In the case of the screw (p. 31), it is evident that while the power describes the circumference of a circle, the weight or resistance is moved through a distance equal to the interval between the threads of the screw, and consequently, by equation (5),

Power × circumference of circle through which it moves = Resistance × interval between the threads.

Or,

The power is to the resistance as the interval between the threads is to the circumference of the circle described by the power.—Q. E. D.

6. **The Pulley.**—Whatever be the complexity of the system of pulleys considered, the principle of constancy of work done enables us to determine the ratio of the power to the resistance; for we have simply to multiply the power by the space through which it passes, and to equate this product to the resistance multiplied by the space through which it is lifted in the same time.

Thus, in the single moveable pulley (p. 34), the power P will evidently descend through a space twice as great as that through which the resistance R is lifted, and, therefore, the power is half the resistance.

In the system (p. 35) the power will descend through six times the height to which the weight R is raised, and, therefore, the power is only one-sixth of the weight lifted; and, in a similar manner, the ratio of the power to the weight lifted may be found in other cases.

7. **The Gain in Power is Loss in Time.**—The principle of the constancy of work done by a force, which has been illustrated by the cases of the common mechanical powers, is of universal application (always neglecting friction), and may be used to find the ratio of the power to the resistance in the most complicated as well as in the simplest machines; and it possesses this practical advantage over all other methods of investigation, that it only requires us to know the spaces traversed by any two points of the machine in the same time, to be

able to state the ratio of the forces acting at these points. The spaces described by the different parts of a machine are found by direct experiment, and easily measured, and hence this principle, although under a different name, has been long familiar to practical mechanics. It is generally stated in the following form, that "what is gained in power is lost in time," or, in other words, that all parts of any machine do the same amount of *work*, because if any point moves with a greater force than another, it must move slower, in such a manner as that the product of the force into the space it moves through in a given time shall be the same for this point as for all other points of the machine. In every case of machinery, however complicated; the equation

$$Pp = Rr, \qquad (5)$$

is always true, as explained in page 39.

DYNAMICS.

CHAPTER I.

DEFINITIONS AND LAWS OF MOTION.

1. Motion, or Velocity.—2. Quantity of Matter and Motion.—3. Laws of Motion.

DYNAMICS is that branch of Mechanics which treats of motion, considered as an effect of force. In Statics, we compared forces by means of the pressures produced by them; in Dynamics, we are to compare forces by means of the motions produced; the first thing to be done is, therefore, to explain what is meant by motion, or velocity.

1. **Motion, or velocity.**—The motion or velocity of a body is the rate at which the body is moving, and is measured by the space passed over in a given time; such as miles per hour, feet per second, &c.

In order to have a uniform unit of velocity, we shall invariably express velocities in feet and seconds; and when velocities are expressed in any other way, they must be reduced to their equivalent value in feet and seconds. The unit velocity is, therefore, the velocity of a body which is moving at the rate of one foot in a second. Velocities are either uniform or variable. A uniform velocity is one in which the velocity is constant at each instant; a variable velocity is one in which the velo-

city, measured at different instants, is different. The velocity of a body, at any instant, is measured by finding the number of feet which would be described in a second by a body supposed to move uniformly at the rate at which the given body is moving at the given instant.

Let V represent the velocity of a body moving uniformly; i. e. the number of feet it describes in one second; and let S denote the number of feet passed over in T seconds. Then,

$$V : S :: 1 : T;$$

i. e., the space described in one second : the space described in T seconds : : 1 second : T seconds. Hence (since the rectangle under the means is equal to the rectangle under the extremes),

$$S = VT. \qquad (1)$$

This equation contains three quantities—space, time, and velocity; any two of these being given, we can calculate the third.

EXAMPLES.

1. A body moves at the rate of 754 yards per hour. Find the velocity in feet per second. *Ans.* 0.628 ft.

2. A railway train travels at the rate of 40 miles per hour. Find its velocity in feet per second. *Ans.* 58.66 ft.

3. A train takes 7 h. 31 m. to travel 200 miles. Find its velocity. *Ans.* 39.02 ft.

4. Supposing the diameter of the earth at the equator to be 7925 miles, and that it revolves once in 23 h. 56 m., find the velocity of a point at the equator. *Ans.* 1525.7 ft.

5. Sound travels at the rate of 1090 feet per second. If a gun be fired at Holyhead, how long will the sound take to reach Dublin, the distance being 64 miles? *Ans.* 5 m. 10 s.

6. A railway train sets out to perform a journey of 347 miles, and travels at the rate of $30\frac{1}{2}$ miles per hour; after travelling 261 miles an accident happens to the machinery, which stops the train 2 h. 27 m., and renders the locomotive incapable of travelling at the rate of more than 16 miles per hour. Find the time occupied by the whole journey. *Ans.* 16 h. 26 m.

7. A steamer going at the rate of 12 knots per hour steams for 4 days, and then alters her velocity to 13 knots per hour, at which she continues for 3 days 7 hours. Find the distance in miles she has travelled at the end of the whole time.

Ans. 2507 miles.

Note.—A knot, or geographical mile, is the sixtieth part of a degree of the meridian, and contains 6076 ft. in latitude 45°.

Having defined velocity or motion, we can now explain the mode of estimating a force *dynamically;* in statics we estimated the magnitude of a force by the number of pounds or ounces it was capable of supporting; in dynamics we estimate the magnitude of a force by the *velocity it is capable of producing in a given time*, such as a second. We suppose the force to act constantly upon the body originally at rest, for the space of one second, and then to cease; the rate or velocity at which the body is then moving is the dynamical measure of the force.

2. **Quantity of matter and of motion.**—The following definitions should be carefully committed to memory.

DEFINITION.—The *specific gravity*, or density of a body, is *the number which expresses the ratio which the weight of a cubic inch of the body bears to the weight of a cubic inch of distilled water, at the temperature of* 60° *Fahrenheit.*

DEFINITION.—*The quantity of matter* contained in a body is the *product of its specific gravity and*

volume expressed in cubic inches, feet, &c. The quantity of matter is also called the *mass of a body*.

DEFINITION.—*The quantity of motion* in a body is the product of its *quantity of matter*, *and the velocity with which all its parts are moving*. The quantity of motion is also called the *momentum of a body*.

The method of finding the specific gravities of bodies, and a Table of specific gravities, is given in the Manual of Hydrostatics. Here it is sufficient to give the definition of specific gravity, and to add a few examples to illustrate the definitions of quantity of matter and motion.

EXAMPLES.

1. The specific gravity of lead and copper being 11.35 and 8.90, respectively (i. e. the weight of a cubic inch of water at 60° F. being unity, the cubic inch of lead and copper will weigh respectively 11.35 and 8.90), find the ratio of the quantities of matter contained in a cubic foot of solid lead and a cubic yard of solid copper. *Ans.* 0.04723.

2. The specific gravities of hammered gold and silver being 19.35 and 10.51, find the ratio of the quantities of matter contained in a bar of gold measuring 4.17 inches long, by 0.64 inches wide, by 0.31 inches deep; and a bar of silver measuring 13.22 inches long, by 1.14 inches wide, by 0.65 inches deep. *Ans.* 0.15549.

3. A bar of solid silver (sp. gr. 10.51) moves with a velocity of 5 miles per hour, its dimensions being the same as in the last example. Find its quantity of motion, the volume being measured in cubic inches. *Ans.* 755.012.

4. A cubic foot of copper (sp. gr. 8.90) moves with a velocity of 1407 yards per minute. Find its quantity of motion, volume being measured in cubic inches. *Ans.* 1081926.72.

5. A block of Carrara marble (sp. gr. 2.716), measuring 2 yards long, by 2 feet wide, by 1.16 feet deep, in falling from a height acquires a velocity of 13 feet per second. What is its *momentum*, volume being measured in cubic feet. *Ans.* 491.48.

6. What should be the velocity of a cubic foot of lead (sp. gr. 11.35), in order that its momentum should be equal to that of the marble block in the last example?

Ans. 43.302 feet per second.

7. A bullet of lead, containing 0.267 cubic inches, is projected from a gun. Find with what velocity it should move, in order that its momentum should be equal to the momentum of a ball of copper, containing 13.47 cubic inches, moving at the rate of 14 feet per minute. *Ans.* 9.230 feet per second.

3. **Laws of motion.**—The laws of motion are simple statements of observed facts, from which can be deduced, without any further appeal to experiment, the laws of the most complicated motions. They are summed up by Sir Isaac Newton in three laws, with the first two of which only we are here concerned. These two laws of motion are the following:—

LEX I.

"*Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.*"—Princip. Math. Jes. Edit. tom. i. p. 15.

A body will continue in its state of rest, or state of uniform motion in a right line, unless compelled to alter that state by force impressed upon it.

This law expresses the fact that matter is indifferent to rest or motion; or that a state of uniform rectilinear motion is as much a proof as a state of rest that a body is not acted upon by force; or rather, that the forces acting upon it equilibrate each other. One of the best illustrations of this fundamental property of matter is derived from the facts observed in driving a train of carriages on a horizontal railroad; it is found that it requires a

constant expenditure of force, i. e. of coals or steam, to maintain a constant velocity. In this case the forces which tend to destroy the motion of the train are friction, and the resistance of the air, both of which are constant at a constant velocity; when these forces are equilibrated or destroyed by a constant expenditure of propelling force, the train is in the same condition as if it were acted upon by no forces whatever, and therefore, in accordance with the first law of motion, continues to move uniformly forward, until stopped by some external force. The property of matter which is described in this law of motion, viz., its incapacity to alter its condition, whether of rest or of motion, is frequently called the *inertia of matter.*

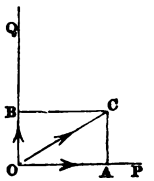
LEX II.

"*Mutationem motus proportionalem esse vi motrici impressæ, et fieri secundum lineam rectam qua vis illa imprimitur.*"—Princip. Math. Jes. Edit., tom. i. p. 15.

The change in the quantity of motion produced is proportional to the force applied, and takes place in the direction of that force.

This law of motion contains three distinct statements, which, for greater clearness, we shall give separately; the first relates to direction, the second and third to magnitude.

FIRST STATEMENT.—*The motion produced in a body by the application of a force takes place in the direction of the force, and is independent of any previous motion existing in the body.* The fact expressed by this proposition is generally called the *Law of the Composition of Velocities*, and may be thus stated:



If a body placed at O move in the direction OP, with a velocity which would bring it to the point A in one second; and at the same time move in the direction OQ with a velocity which would bring it to the point B in one second; the effect of both velocities acting jointly will be to bring the body, at the end of one second, to the point C, found by completing the parallelogram OACB. The truth of this fundamental law of motion is exhibited in many familiar cases; for example, on board of a ship moving equably, the movements of the sailors are not influenced by the motion of the ship; it is certain, also, from the facts observed by astronomers, that the earth is in motion, nevertheless, all motions take place on its surface independently of that motion, and as if it were at rest.

It appears, therefore, that *Velocities* are compounded by the same law as *Pressures*, and that all the results contained in Chap. I. sec. 4, of Statics, may be transferred to Dynamics, by substituting Velocities for Pressures.

This identity in the Laws of Composition of Velocities and Pressures is only natural; for since Velocity and Pressure are only different effects of the same Force, it is natural to expect that, to a great extent, they will be governed by the same laws.

EXAMPLES.

1. A body moves under the influence of two velocities, at right angles to each other, equal respectively to 17.14 feet and 13.11 feet per second. Find the magnitude of the resultant motion.

Ans. 21.579 feet per second.

2. Find in the same case the angles into which the resultant divides the right angle. *Ans.* 37° 25′ and 52° 35′.

3. A ship sails due north, at the rate of 4 knots per hour, and a ball is rolled towards the east, across her deck, at right angles to her motion, at the rate of 10 feet per second. Find the magnitude of the real motion of the ball, compounded of these two motions.
Ans. 12.064 feet per second.

4. Find the direction of the real motion. *Ans.* 56° East of N.

5. A body moves, under the influence of two velocities, at the rate of 36.14 feet per second, and one of the component velocities is 14.31 feet per second. Find the other, on the supposition that they act at right angles. *Ans.* 33.18 feet.

6. If the component velocities be 14.37 feet and 19.22 feet, and make an angle of 22°, find resultant velocity. *Ans.* 33 feet.

7. Find the angles into which the resultant in the last example divides the angle between components. *Ans.* 12° 36′ and 9° 24′.

8. The component velocities are 13.61 feet per second, and 14.62 yards per minute, and make an angle of 41°. Find magnitude of resultant velocity. *Ans.* 14.07 feet.

9. Find the angles into which it divides the angle between components. *Ans.* 39° 3′ and 1° 57′.

10. A boat is rowed across a river (which is $1\frac{1}{4}$ miles wide) in a direction making an angle of 87° with the bank. The boat travels at the rate of 5 miles per hour, and the river runs at the rate of 2.3 miles per hour. Find at what point of the opposite bank the boat will land, if the angle of 87° be made against the stream.
Ans. 898 yards down the stream from the opposite point.

11. In the same case, if the angle of 87° be made with the stream, find the point of landing.
Ans. 1129 yards down the stream from the opposite point.

12. A ship sails E.N.E. at the rate of 5 knots per hour, and is drifted by the tide E. by S. at the rate of 3 knots per hour. Find her real motion in magnitude. *Ans.* 7.68 knots per hour.

13. Find the direction of her real motion.
Ans. 80° 3′ E. of N.

14. In Example 10, at what angle with the bank should the boat be rowed, in order to land at the opposite point of the river.
Ans. 62° 36′ against the stream.

Besides the statement relating to direction which has been just explained, the second law of motion contains the two following statements relating to magnitude.

SECOND STATEMENT.—*The velocities produced in the same body by different statical forces acting for one second, are proportional to those forces.*

THIRD STATEMENT.—*If the same statical force be applied to move different bodies, the velocities produced in one second will be inversely proportional to the quantities of matter in the bodies moved.*

Both these statements may be expressed algebraically in a single formula, as follows:—Let F represent the Statical force in pounds, ounces, &c., and let f represent the Dynamical force, i. e. the velocity of the body at the end of one second, expressed in feet per second; and let m represent the Quantity of matter in the body moved.

The equation,

$$F = mf, \qquad (2)$$

expresses that the Dynamical force f is directly proportional to the Statical force F, and inversely proportional to the Quantity of matter m; or that *the Statical force applied to move a body is directly proportional to the Quantity of motion produced in a given time.*

The meaning of the second and third Statements just given may, perhaps, be rendered more clear by the following illustration:—

If a force equal to 1 lb. be applied to move a body which contains a quantity of matter represented by unity, it will produce a velocity of 1 foot per second in one second; if a force of 2 lbs. be applied, it will produce a velocity of 2 feet per second;

if 3 lbs., 3 feet per second; and so on: this is the second statement.

If a force of 1 lb. be applied to produce motion in a body whose quantity of matter is represented by unity, in one second it will produce a velocity of 1 foot per second; if it be applied to move a body containing twice the quantity of matter, the velocity produced will be only $\frac{1}{2}$ ft. per second; if the quantity of matter be three times as great, the velocity will be $\frac{1}{3}$ ft. per second, and so on: this is the third statement.

EXAMPLES.

In equation (2), f is understood to be expressed in feet; but the units to which F and m are referred, although arbitrary, are so connected, that one of the units being fixed, the other is also determined.

Let us suppose, for example, that we select for the unit of matter the quantity of matter contained in a cubic inch of distilled water at a temperature of 60° Fahrenheit, then the unit of weight to which F is referred may be thus found. The unit of F is the force which will produce in one second a velocity of one foot per second in a cubic inch of distilled water; but it is found from experiment that if a cubic inch of water be allowed to fall in vacuo at Dublin, its weight, i. e. the force due to terrestrial gravity which acts upon it, will produce in one second a velocity of 32.1948 feet per second; hence,

Unit of F: weight of cubic inch of water : : 1 : 32.1948.

Or, finally—

$$\text{Unit of } F = \frac{\textit{weight of cubic inch of water}}{32.1948}.$$

If the weight of the cubic inch of water be expressed in grains, the unit of F will therefore be 7.842 grs.; because the weight of a cubic inch of water is 252.5 grs.; in other words, it would require a force of 7.842 grs. to produce in a cubic inch of distilled water at 60° F. a velocity of 1 ft. per second. Hence, if in equation (2) m be referred to cubic inches of water, F must be expressed in the units just found. If, on the other hand, the unit of F be given, the unit to which m is referred may be found as

follows:—Let us suppose that F is expressed in pounds avoirdupois: from what has been already stated, it appears that a force of one pound would produce a velocity of 32.1948 feet in the quantity of matter contained in a pound weight of water, hence it would produce a velocity of one foot per second in the quantity of matter contained in 32.1948 lbs. weight of water, i. e. in the quantity of matter contained in 892.5 cubic inches of distilled water at the temperature 60° F. (because the pound avoirdupois contains 7000 grains). The unit of volume to which m is referred is, therefore, in this case, 892.5 cubic inches.

1. If the unit of weight in equation (2) be one grain, find the corresponding unit of volume to which m is referred.

Let x denote the required unit of volume, expressed in cubic inches; then

$$x \times 252.5 \text{ grs.} : 1 \text{ gr.} :: 32.1948 \text{ ft.} : 1 \text{ ft.}$$

Ans. 0.127 cubic in.

2. If the unit of weight be one ton, express the corresponding unit of m in cubic yards of water at 60° Fahrenheit.

Let x denote the required unit of volume, expressed in cubic feet; then, since a cubic foot of water is 62.32 lbs.,

$$x \times 62.32 \text{ lbs.} : 2240 \text{ lbs.} :: 32.1948 \text{ ft.} : 1 \text{ ft.}$$

Hence $x = 1157.2$ cub. ft.

Ans. 42.858 cub. yds.

3. If the unit of matter be the quantity of matter contained in a cubic foot of water, find the corresponding unit of weight.

Let x denote the unit of weight in lbs.; then

$$x : 62.32 \text{ lbs.} :: 1 \text{ ft.} : 32.1948 \text{ ft.}$$

Ans. 1.935 lbs.

4. If the unit of matter be the quantity of matter contained in a cubic inch of lead (sp. gr. = 11.35); find the corresponding unit of weight.

Let x denote the unit of weight expressed in grains; then

$$x : 11.35 \times 252.5 \text{ grs.} :: 1 \text{ ft.} : 32.1948 \text{ ft.}$$

Ans. 89.001 grs.

5. If a force equal to 3 lbs. produce in one second a velocity of 0.317 ft. in a given body; find the quantity of matter contained in the body.

Let x denote the number of cubic feet of water, which contain the same quantity of matter as that in the body; then

$$x \times 62.32 \text{ lbs.} : 3 \text{ lbs.} :: 32.1948 \text{ ft.} : 0.317 \text{ ft.}$$

Ans. Quantity of matter contained in 4.883 cub. ft. of water.

6. If a force of 13.16 lbs. produce in a cubic foot of matter, in one second, a velocity of 4.16 ft., find the specific gravity of the matter.

Let x be the required specific gravity; then

$$x \times 62.32 \text{ lbs.} : 13.16 \text{ lbs.} :: 32.1948 \text{ ft.} : 4.16 \text{ ft.}$$

Ans. 1.634.

7. If a velocity of 42.310 ft. be produced in one second, in a cubic inch of gold (sp. gr. = 19.35); find the magnitude of the force required to produce this velocity.

Let x denote the required force expressed in grains; then

$$x : 19.35 \times 252.5 \text{ grs.} :: 42.31 \text{ ft.} : 32.1948 \text{ ft.}$$

Ans. 64.21 grs.

8. If a force of 17 lbs. produce a velocity of 14 ft. in a cubic foot of matter, in one second, find its specific gravity.

Let x denote the required specific gravity; then

$$17 \text{ lbs.} : x \times 62.32 \text{ lbs.} :: 14 \text{ ft} : 32.1948 \text{ ft.}$$

Ans. 0.627.

The laws of motion which have been given in this chapter have been tested by the most careful and accurate experiments; but from their very nature they do not admit of any very direct mode of proof. The learner must be satisfied to take them for granted at the commencement, and he will find that, as he progresses in the study of Dynamics, the proofs of the truth of the fundamental laws will multiply; for since the whole science is based upon these laws, every complicated deduction from them, which is verified by experience, will afford a fresh proof of the truth of the laws themselves, and thus furnish evidence which, though *indirect*, is as certain as direct evidence.

As an illustration of this kind of proof, we may mention the prediction of the time and circumstances of an eclipse of the sun or moon; this prediction is a deduction from the elementary laws of

motion, and its exact agreement with the observed facts of the eclipse is obviously a strong proof of the truth of those laws of motion from which the prediction is derived.

Although, in order to follow Newton's method, we have given but two laws of motion, it is plain that these are equivalent to *four distinct* laws, as has been explained already. The student is requested to commit to memory the three statements contained in the second law of motion, together with the two laws of motion, before he proceeds to he study of the next chapter.

CHAPTER II.

ON RECTILINEAR MOTION.

1. Relation between Velocity and Time.—2. Relation between Space and Time.—3. Relation between Velocity and Space.—4. Motion of Falling Bodies.—5. Motion of Bodies on Inclined Planes.—6. Experimental Proofs of the Laws of Rectilinear Motion.

1. **Relation between velocity and time.**—We shall, in this chapter, investigate the laws of motion of bodies acted upon by constant forces, and moving in right lines. The dynamical force is supposed to be known, and the body is supposed to set out from a state of rest. The first thing to be done is to determine the velocity produced in a given time.

Let f represent the Dynamical force, i. e. the velocity acquired at the end of one second, and let

v represent the velocity at the end of t seconds. Then,

$$v : f :: t : 1,$$

i. e. the velocity acquired in t seconds is to the velocity acquired in 1 second as $t : 1$; hence,

$$v = ft. \qquad (3)$$

This equation expresses the relation between the velocity, v, the Dynamical force, f, and the time, t, during which it acts, and enables us from any two of these quantities to calculate the third.

EXAMPLES.

1. A velocity of 14 feet per second is produced by a force acting for 3 minutes. Calculate the force.

Ans. 0.0777 ft. per second.

2. A force acting for 13 sec. produces a velocity of 4 miles per hour. Find its magnitude. *Ans.* 0.451 ft.

3. A force equal to 7 feet per second acts for 3 minutes. Find the velocity produced. *Ans.* 1260 ft. per second.

4. A force equal to 32.195 feet per second acts for 5 seconds. Find the velocity produced. *Ans.* 160.975 ft.

5. Find the time during which a force of 11.16 feet must act to produce a velocity of 100 feet per second. *Ans.* 8.96 seconds.

6. During what time must a force of 32.195 feet act to produce a velocity of 375 feet per second? *Ans.* 11.64 seconds.

2. **Relation between space and time.**—Let us suppose the velocity of the body acted on by the constant force to increase *per saltum*, and to receive equal increments of velocities in equal intervals of time. If the final velocity be v, and the whole time t be divided into n equal intervals, it is evident that the velocity during the first interval is $\frac{v}{n}$, during

the second, $\frac{2v}{n}$, during the third, $\frac{3v}{n}$, and so on.

And since the space described by a body moving uniformly is equal to the product of the velocity and time, page (45), it follows that

Space described in first interval $= \frac{v}{n} \times \frac{t}{n} = \frac{vt}{n^2}$,

" second " $= \frac{2v}{n} \times \frac{t}{n} = \frac{2vt}{n^2}$,

" third " $= \frac{3v}{n} \times \frac{t}{n} = \frac{3vt}{n^2}$,

" n^{th} " $= \frac{nv}{n} \times \frac{t}{n} = \frac{nvt}{n^2}$.

If s be the whole space described, it must be equal to the sum of the spaces described in the successive intervals, and therefore, adding, we find

$$s = \frac{vt}{n^2} \times \{1 + 2 + 3 + 4 + \&c. + n\};$$

but the series included within the parenthesis is equal to $\frac{n(n+1)}{2}$ (*vid.* Manual of Algebra), therefore

$$s = \frac{vt}{2} \frac{n(n+1)}{n^2} = \frac{vt}{2}\left(1 + \frac{1}{n}\right).$$

If now we suppose the velocity to increase, not *per saltum*, but *continually* and in proportion to the time, we must suppose the intervals of time to be indefinitely small, i. e. the number of intervals n indefi-

nitely great; on which supposition $\frac{1}{n} = 0$, and therefore

$$s = \frac{vt}{2}.$$

Now if a body be acted on by a constant force, the velocity increases in proportion to the time, equation (3), and therefore, *If a body move from rest under the action of a constant force, the space described is equal to half the product of the time and the final velocity.*

In this case, as $v = ft$ by equation (3),

$$s = \frac{ft^2}{2}. \qquad (4)$$

This equation contains three quantities, viz., s the space, t the time, and f the Dynamical force. Given any two of these, we can calculate the third.

EXAMPLES.

1. If a body, under the influence of a constant force, move through 5 feet in 3 seconds, find the magnitude of the force which animates it. *Ans.* 1.111 ft.

2. If the space described in 10 seconds be 300 yards, find magnitude of force. *Ans.* 18 ft.

3. If a body be acted on by a force of 11 feet, during three minutes, find the space described in that time. *Ans.* 178200 ft.

4. If the force be 32.19 feet, and the time 7 seconds, find space described. *Ans.* 788.655 ft.

5. Find the time occupied by a body, acted on by a force of 12.17 feet, in describing 137 yards. *Ans.* 8.21 secs.

6. A body is moved by a force of 32.18 feet, which acts during 11 seconds. Find space described. *Ans.* 1946.89 ft.

3. **Relation between velocity and space.**—The velocity acquired by a body, acted on by a constant force, in describing a given space, may be found from equations (3) and (4), as follows:

Squaring equation (3), we find

$$v^2 = f^2 t^2;$$

multiplying equation (4) by $2f$, we find

$$2fs = f^2 t^2;$$

hence,

$$v^2 = 2fs. \qquad (5)$$

This equation contains three quantities, viz., f the Dynamical force, s the space described, and v the velocity acquired. Any two of these being given, we can calculate the third.

EXAMPLES.

1. Find the force which would cause a body to acquire a velocity of 35 feet per second, after describing 100 yards.
Ans. 2.04 ft.

2. If the space described be one mile, and the velocity acquired be 1142 yards per minute, find force. *Ans.* 0.309 ft.

3. A body acted on by a force of 13 feet per second describes 1100 feet. Find its velocity. *Ans.* 169.1 ft.

4. If the force be 32.190 feet, and the space described be 467 feet, find velocity acquired. *Ans.* 173.4 ft.

5. With the same force, let the velocity acquired be 75 feet per second. Find space described. *Ans.* 87.3 ft.

6. If a body, animated by a force of 314 yards per minute, acquire a velocity of 411 feet per second, find space described.
Ans. 5379.6 ft.

4. **Motion of falling bodies.**—The most important of all the forces with which we are acquainted is the force of Gravity, which, considered as a Dynamical force, is the cause of the motion of falling bodies. This force, as has been mentioned already, in the chapter on the Centre of Gravity, varies from place to place on the surface of the globe. The Table which is given in page 64 contains the values of the Dynamical force of gravity in English feet and decimals, reduced to the sea level, at some of the most important places on the earth's surface. It has been calculated from pendulum experiments in a manner which cannot be described here. It is easy to see from it that *at all places on the surface of the earth, the velocity acquired in one second by a body falling in vacuo is somewhat greater than* 32 *feet per second.*

The force of gravity is so important that it is usual in works on Mechanics to designate it by a special letter g, which in future we shall always use to denote the Dynamical force of gravity, expressed in feet.

If W denote the weight of a body, and m its mass, equation (2), when applied to the force of gravity, will become

$$W = mg. \qquad (6)$$

This equation expresses the fact, that the weight of a body at different places on the earth's surface is proportional to the velocity acquired by it in falling for one second, and is an instance of the truth of the second law of motion.

Equations (3, 4, 5), referred to the force of gravity, become

$$v = gt, \tag{7}$$

$$s = \tfrac{1}{2}gt^2, \tag{8}$$

$$v^2 = 2gs. \tag{9}$$

These three equations contain the complete solution of all problems relating to falling bodies.

If the time be made equal to 1 second in equation (7), we find

$$v = g,$$

i. e. the velocity acquired in one second is the measure of the force of gravity.

If in equation (8), we make $t = 1$ sec., we obtain

$$s = \tfrac{1}{2}g,$$

i. e. *the space described by a falling body in one second is half the velocity acquired in one second, or equal about 16 feet.*

If we now make t, in equation (8), successively equal to 2, 3, 4, &c., seconds, we find

$s = \frac{1}{2}g \times 4 = 16 \times 4$ ft. = 64 ft. in 2 seconds,

$s = \frac{1}{2}g \times 9 = 16 \times 9$ ft. = 144 ft. in 3 „

$s = \frac{1}{2}g \times 16 = 16 \times 16$ ft. = 256 ft. in 4 „

i. e. *the spaces described from the commencement of the motion vary as the squares of the number of seconds.*

If we subtract the space described in one second from the space described in two seconds, i. e. 16 feet from 64 feet, we shall obtain the space described in the second second, or 48 feet; and in like manner, if we subtract the space described in two seconds from the space described in three seconds, we should find the space described in the third second, and so on. We thus find,

The space fallen through in the			first second	=	16 ft.
,,	,,	,,	second ,,	=	48 ft.
,,	,,	,,	third ,,	=	80 ft.
,,	,,	,,	fourth ,,	=	112 ft.
,,	,,	,,	fifth ,,	=	144 ft.
&c.			&c.		&c.

It is easy to see that these numbers are *once*, *three* times, *five* times, *seven* times, &c. 16 feet. Hence, *the spaces described by a falling body in successive seconds are proportional to the series of odd numbers.*

EXAMPLES.

1. Find the space described in the *ninth* second by a falling body. *Ans.* 272 ft.

2. Find the space described in *nine* seconds by a falling body. *Ans.* 1296 ft.

3. Find the space described between the *fourth* and *seventh* second by a falling body. *Ans.* 528 ft.

4. Find the space described between the *third* and *eleventh* second. *Ans.* 1792 ft.

Equation (8) enables us to calculate the height of any place approximately, by counting the time taken by a body (such as a stone) to fall from the top to the bottom. The simplest rule for this rough calculation is the following:—

The height of any place in feet is equal to the square of the number of quarter seconds occupied by a body in falling from the top to the bottom.

Since many watches are constructed to beat quarter seconds, the observation may be made by the ear, and the height will be given by the square of the number of beats of the watch during the descent of the body.

This rule is proved as follows:—The space described is equal, by equation (8), to $\frac{1}{2}g$, or 16 ft., multiplied by the number of seconds squared, or by the square of the quarter seconds divided by the square of 4; hence the 16 destroys in the numerator and denominator; leaving the height equal to the square of the number of quarter seconds.

Equations (7, 8, 9) contain the three quantities, space, time, and velocity, any two of which being given, the third may be calculated, provided the force of gravity be known at the given place.

TABLE OF THE DYNAMICAL FORCE OF GRAVITY.

Station.	Latitude.	Force of Gravity at Sea Level.	Names of Observers.
		FEET.	
1. London,	51° 31′ 8″ N.	32.1908	Sabine.
2. Paris,	48 50 14 N.	32.1820	Borda.
3. Dunkerque,	51 2 10 N.	32.1895	Biot, Mathieu.
4. Bordeaux,	44 50 26 N.	32.1691	Biot, Mathieu.
5. Leith,	55 58 41 N.	32.2040	Kater.
6. Rio de Janeiro,	22 55 13 S.	32.1121	Freycinet.
7. Cape of Good Hope,	33 55 15 S.	32.1403	Freycinet.
8. Isle of France,	20 9 19 S.	32.1147	Duperrey.
9. Port Jackson,	33 51 39 S.	32.1412	Duperrey.
10. Isle S. Thomas	0 24 41 N.	32.0930	Sabine.
11. Isle Rawak,	0 1 34 S.	32.0881	Freycinet.
12. Ascension,	7 55 48 S.	32.0959	Sabine.
13. Sierra Leone,	8 29 28 S.	32.0927	Sabine.
14. Trinidad,	10 38 56 N.	32.0917	Sabine.
15. Jamaica,	17 56 7 N.	32.1050	Sabine.
16. New York,	40 42 43 N.	32.1600	Sabine.
17. Greenland,	74 32 19 N.	32.2435	Sabine.
18. Spitzbergen,	79 49 58 N.	32.2526	Sabine.
19. Unst (Shetland),	60 45 25 N.	32.2173	Biot.
20. Toulon,	43 7 9 N.	32.1668	Duperrey.

EXAMPLES.

1. Calculate the velocity acquired in 11^s by a body falling in Ascension Island. This problem is solved by equation (7); by the Table, $g = 32.0959$ ft., which multiplied by 11 is equal to the required velocity, 353.0549 feet.

2. Calculate the space described in 15^s by a body falling in Jamaica. By equation (8) the space is equal to half the force of gravity multiplied by the square of the time; but by the Table, $g = 32.1050$ ft., hence $\frac{1}{2}g = 16.0525$, which multiplied by 225 is equal to 3611.8125 ft.

3. Calculate the velocity acquired by a body in falling through 100 ft. in Spitzbergen. By equation (9) the square of the velocity is equal to the force of gravity multiplied by twice the space, i. e. 200 ft. multiplied by $g = 32.2526$, equal 6450.52; extracting the square root we find $v =$ 80.315 ft.

4. Calculate the space described by a falling body at Port Jackson, in $3^s.151$. *Ans.* 159.5617 ft.

5. Calculate the time required to fall 316 ft. in New York. *Ans.* 4.43 secs.

6. Find the velocity acquired by a falling body in Paris, in 9 seconds. *Ans.* 289.638 ft.

7. If a body falls in Sierre Leone through 216.171 ft.; find the velocity acquired. *Ans.* 117.79 ft.

8. Find, in the same case, the time required to describe this space. *Ans.* 3.67 secs.

9. If a body falling at the Cape of Good Hope acquire a velocity of 71.30 ft.; find the space described. *Ans.* 79.085 ft.

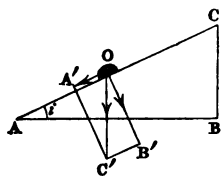
10. Find also the time of describing this space. *Ans.* 2.21 secs.

11. Find the time required by a falling body in Bordeaux to acquire a velocity of 131.17 ft. *Ans.* 4.077 secs.

12. Find the time required by a body to fall 200 ft. in Trinidad. *Ans.* 3.530 secs.

5. **Motion of bodies on inclined planes.**—If a body, instead of falling in a vertical line, be

allowed to run down an inclined plane, the force which moves it may be varied at pleasure, by varying the inclination of the plane to the horizon. The magnitude of the force which impels a body down an inclined plane may be thus found:—Let AC be an inclined plane, whose inclination BAC is denoted by the letter i; let O be a body placed upon this plane; draw the vertical line OC′ equal to g, the gravity at the place, and complete the parallelogram C′A′OB′; OA′ is the component of gravity which moves the body down the plane; we shall call it f; OA′ is equal to OC′ multiplied by the sine of OC′A′, which is equal to BAC or i, because the triangles OC′A′ and BAC are similar; hence—



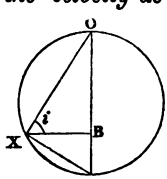
$$f = g \sin i. \qquad (10)$$

This equation enables us, by knowing the amount of gravity, and the inclination of the plane, to solve all problems respecting the motion of bodies on inclined planes by means of equations (3, 4, 5).

There are two general properties of the motion of bodies on inclined planes which it is necessary to prove—

1st. *The velocity acquired by a body in running down any inclined plane is equal to the velocity acquired in falling down the height of the plane.*

2nd. *If a circle be drawn in a vertical plane, and from its highest point* O, *chords be drawn; the time occupied by a body in running down any chord is constant.*



The first of these propositions is thus proved: let h represent the height of the plane, i. e. the line BC (fig. p. 66); by equation (5), $v^2 = 2fs$; substituting for f its value $g \sin i$ (10), we find $v^2 = 2gs \sin i$; but $s \sin i$ is equal to h, since s is equal to AC; therefore, finally, $v^2 = 2gh$.

Or $$v = \sqrt{2gh}.$$

This is the velocity acquired by a body in running down from C to A; but the velocity acquired in falling from C to B under the influence of gravity is also equal to $\sqrt{2gh}$ by equation (9); consequently the velocity acquired in running down any inclined plane is equal to the velocity acquired in falling down its height.—Q. E. D.

The second proposition may be proved as follows: The time of running down any chord OX is, by equation (4),

$$t = \sqrt{\frac{2s}{f}},$$

where s is equal to OX and $f = g \sin i$; i being the angle OXB, which is equal to OAX, because OAX is a right-angled triangle, and XB is perpendicular to OA; if we represent the diameter OA by h, we shall have OX equal to OA multiplied by the sine of OAX; or, $s = h \sin i$; substituting in the equation already found these values of f and s, we obtain finally—

$$t = \sqrt{\frac{2h}{g}}.$$

This is the time of running down the chord OX; but by equation (8) the time of falling down the

diameter OA is also represented by $\sqrt{\frac{2h}{g}}$; hence the time of running down any chord is equal to the time of falling down the diameter, and is therefore constant.—Q. E. D.

EXAMPLES.

1. Find the time occupied by a body in running down 1000 ft. on an inclined plane at Dunkerque, the inclination of the plane being 1°. The time of descent, by equations (4) and (10), is $\sqrt{\frac{2s}{g \sin i}}$, which is equal to $\sqrt{\frac{2000}{0.5617}}$, because $s = 1000$, $g = 32.1895$, $\sin i = 0.01745$; extracting the square root we find $t = 59.67$ secs.

2. If the inclination of a plane at Toulon be 36′, find the velocity acquired in running down 1000 ft. By equations (5) and (10), $v = \sqrt{2gs \sin i} = \sqrt{673.57}$, since $g = 32.1668$, $s = 1000$, and $\sin i = 0.01047$. Extracting the square root we find $v = 25.95$ feet.

3. Find the velocity acquired in 10^s by a body running down an inclined plane in the Island of Ascension, whose inclination is 17′. By equations (3) and (10) we find $v = gt \sin i$, which, since $g = 32.0959$, $t = 10$, and $\sin i = 0.00494$, is equal to 1.585 ft.

4. An inclined plane at Bordeaux has an inclination of 1 foot in 65; find the velocity acquired by a body running down it through a space of 427 ft. *Ans.* 20.558 ft.

5. Find the time of describing 427 ft. *Ans.* 41.54 secs.

6. Calculate the space described in 3^s. *Ans.* 2.227 ft.

7. Find the velocity acquired by a railway train in running down a gradient of 2164 ft., having a total fall of 31 ft.; the force of gravity being 32.19, and the resistance from friction and the air being estimated at 7 lbs. per ton. If f denote the force down the plane, and g the force of gravity,

$$f = g\left\{\frac{31}{2164} - \frac{7}{2240}\right\} = 0.36054 \text{ ft.};$$

and therefore *Ans.* 39.50 ft.

8. Find, under the same circumstances, the time of running down 1000 ft. *Ans.* 74.48 secs.

9. If the inclination of a plane in Greenland be 14°; find the space described in 5 secs. *Ans.* 97.503 ft.

10. Find the space described in 5", on a similar plane, in the Island of St. Thomas, Gulf of Guinea. *Ans.* 97.048 ft.

11. Calculate the velocity acquired, in Ex. (9) in running down 100 ft. *Ans.* 39.5 ft.

12. Also the velocity acquired in 11". *Ans.* 85.80 ft.

6. **Experimental proofs of the laws of rectilinear motion.**—The laws of rectilinear motion of a body, under the influence of a constant force, have been deduced from the laws of motion laid down in the first chapter; but this method, although the most systematic, is not the most natural, since the Laws of Motion were originally deduced from observations made on the motion of bodies acted on by a constant force. The only constant force suitable for such observations, with which we are acquainted, is the force of gravity; but as the velocity produced by this force is too great for direct experiments, it is necessary to devise some method by which the force of gravity shall be so diminished, and the motions produced by it rendered so slow, as to render the observation of them possible. This object has been effected in two ways: one of which is the method employed by Galileo, in his experiments on falling bodies, and the other is the celebrated method used by Mr. Atwood for establishing the elementary laws of Dynamics.

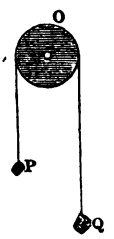
The *Method of Galileo* consists in the use of an inclined plane, or rather inclined cord, about thirty

feet in length, stretched between two points, one of which is higher than the other, on which a weight is made to traverse by means of a small pulley.

It appears from equation (10), that the force which actuates the body is gravity, multiplied by the sine of the angle of inclination; consequently, by increasing or diminishing the inclination, we can increase and diminish the force at pleasure, and so vary the velocity acquired.

Having stretched the cord at a small elevation, if we allow the weight to run down, and note the instant at which the motion commences; by means of a pendulum beating seconds, we can observe the spaces described in the first second, in the first two seconds, in the first three seconds, and so on. This being done, we shall find that the spaces described are proportional to the numbers 1, 4, 9, 16, 25, &c., i. e. to the squares of the times, which establishes the truth of equation (4), and proves that gravity is a constant force. The other laws of rectilinear motion may also be confirmed by the inclined plane of Galileo; but as a means of illustrating the principles of Dynamics, it is very inferior to the method devised by Mr. Atwood, the principle of which admits of an easy explanation, although the details of construction and precautions to be used in observing are unsuited to the limits of this work.

Atwood's Machine, when reduced to its most elementary form, consists of a pulley O (which in practice is generally mounted on four friction wheels), and a cord passing over it, at the two extremities of which are suspended two unequal



weights, P and Q; if these weights were equal there would be no motion, and the smaller the difference between them is, the less will be the motive force and consequent velocity: let f represent the Dynamical force which animates the system; the Statical force which produces this Dynamical force is evidently $P - Q$, i. e. the difference of the weights, and the mass moved is the sum of the masses contained in $P + Q$; if the weights P and Q were allowed to fall together by the force of gravity, they would acquire in one second a velocity equal to g, the Statical force producing this velocity being evidently $P + Q$, i. e. the sum of the weights; we have, therefore, by the second law of motion,

$$f : g :: P - Q : P + Q,$$

because the Dynamical forces are to each other as the Statical forces producing them; hence we obtain

$$f = \frac{P - Q}{P + Q} g. \qquad (11)$$

It is evident that we can make the force f as small as we please by diminishing the difference $P - Q$; the motion produced by this force is therefore completely under our control, and may be made as slow as the convenience of observation requires. All the laws of rectilinear motion caused by constant forces, *vid.* equations (3, 4, 5), may be illustrated by means of this machine, with such accuracy as to leave no doubt of their truth on the learner's mind. Atwood's Machine and the Inclined Plane of Galileo

are both based upon the principle of diminishing the force of gravity, so as to allow of direct observation of the motions produced. So far as *direct* proof of the laws of motion is concerned, no apparatus can be better conceived than Atwood's Machine; but the student must remember that these experiments are here mentioned merely as illustrations, and not proofs of the laws of motion. The most certain proofs of these laws, as we have already mentioned, are to be found in remote and indirect consequences, which admit of easy and accurate observation, and from which we can reason back rigorously to the laws of motion assumed at the commencement of the study of Dynamics.

CHAPTER III.

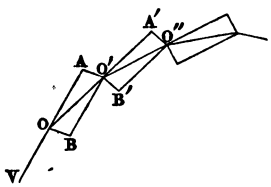
ON CIRCULAR MOTION.

1. Direction and Magnitude of the Deflecting Force.—2. Centrifugal Force.—3. Diurnal Rotation of the Earth.

1. **Direction and magnitude of the deflecting force.**—By the first law of motion it appears that a body in motion, not acted on by any force, will continue to move uniformly in a right line; consequently, any variation in its velocity, or any curvature in its path, is a sure sign that some external force acts upon it.

In the last chapter we considered the alteration of velocity produced by a constant force acting in the direction in which the motion takes place. In

the present chapter we shall consider the mode of estimating the direction and the amount of the force which causes the path of a body to deviate from a right line, and shall apply the calculation to the case of circular motion.



If a body move in the right line VOA *with a velocity such as would cause it to describe the space* OA *in one second, and having arrived at the point* O, *instead of continuing to move in the line* OA, *proceed to describe the line* OO′O″ *in the two following seconds ; it is required to calculate the direction and magnitude of the force which deflects it from its rectilinear path.*

In the first second, the body describes the line OO′, instead of OA; hence (completing the parallelogram OAO′B) OO′ may be considered as the resultant of the velocities OA and OB ; consequently OB is the space described in one second due to the deflecting force; therefore twice the line OB represents the magnitude and direction of this force (*vid.* p. 62). Produce the line OO′ until O′A′ is equal to OO′; if the deflecting force ceased to act, the body would describe the space O′A′ in the next second, and continue to move in the production of this line with uniform velocity. Suppose, however, that at the end of the next second it is actually found at the point O″; completing, as before, the parallelogram O′A′O″B′, the line O′B′ will represent the space described in consequence of the new deflecting force in one second, and, consequently, twice this line will represent the force itself.

We thus see that there exists an intimate connexion between the shape of the path described by a body and the forces which act upon it, and can easily imagine the possibility of solving these two problems.

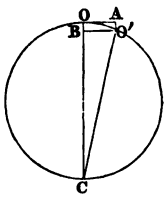
1st. Given the forces acting upon a body, to find the path it will describe.

2nd. Given the path described by a body, to find the forces which act upon it.

The remarkable discovery made by Sir Isaac Newton of the law of gravitation was an instance of the second of these problems. Kepler had proved from Tycho Brahe's observations and his own, that the planets moved in ellipses, having the sun in the focus. Newton, being thus given the shape of the path, discovered the nature of the force which caused the planets to move in that path. Although the principle upon which such calculations are based is very simple, and is contained in the construction just given, yet the actual solution of the problem would require an amount of knowledge which the reader of this work is not expected to possess.

2. **Centrifugal force.**—The simplest example of the principle proved in the last section is the case of a body compelled by external force to describe a circle with uniform velocity; the deflecting force is directed towards the centre, and is equal and opposite to the force with which the body tends to fly off from the centre; this latter force is generally called the centrifugal force.

It is required, being given the velocity of the body, and the radius of the circle described by it, to calculate the magnitude of the centrifugal force.



Let O be the position of the body at any instant, and let OO′ be the space described in the time τ, supposed to be very small; if the body were not acted on by a deflecting force, it would have moved from O to A during the time it actually moves from O to O′; complete the parallelogram OAO′B; the line OB represents the space described under the influence of the deflecting force in the time of describing OO′, and is, therefore, equal, by equation (4), to $\frac{1}{2}f\tau^2$; while the space OO′ is equal, by equation (1), to $v\tau$. The triangle OO′C is a right-angled triangle, OO′C being the angle in a semicircle; therefore OO′ is a mean proportional between OB and OC, and the line OB is equal to the square of OO′, divided by the diameter OC; hence we obtain

$$\mathrm{OB} = \frac{(\mathrm{OO}')^2}{2 \times \textit{radius}}.$$

Substituting for OB and OO′ the values given above, we find

$$\tfrac{1}{2}f\tau^2 = \frac{v^2\tau^2}{2r},$$

and solving for f, we obtain

$$\frac{v^2}{r}. \qquad (12)$$

The centrifugal force, which is equal and opposite to the deflecting force, is therefore directly proportional to the square of the velocity, and inversely proportional to the radius of the circle described.

The equation (12) may be put into a more convenient form, in the following manner. Let T denote the time of describing a complete revolution, called the Periodic Time, and let π represent the ratio of the circumference of a circle to its diameter, i. e. 3.14159; the circumference of the circle described will be $2\pi r$, which is the space described in the time T. By equation (1) the velocity is equal to the space divided by the time, i. e.

$$v = \frac{2\pi r}{T};$$

hence, substituting this value in equation (12), we find

$$f = 4\pi^2 \frac{r}{T^2}; \qquad (13)$$

i. e. the *centrifugal force is directly proportional to the radius of circle described, and inversely proportional to the square of the Periodic time.*

In equations (12, 13), the force f is understood to be the force applied to the unit of mass. If it were necessary to compare the effects of centrifugal force upon different bodies, we should then multiply each centrifugal force by the mass of the body it is applied to move.

EXAMPLES.

1. If a body be compelled to move on a circle with a velocity of 300 yards per minute, the radius of the circle being 16 feet, calculate the centrifugal force. *Ans.* 14.062 ft.

2. If a body be compelled to move on a circle, whose radius is 17 yards, with a velocity of 361 feet per second, calculate the centrifugal force. *Ans.* 2555.3 ft.

3. If a body, weighing 17 tons, move on the circumference of a circle, whose radius is 1110 feet, with a velocity of 16 feet per second, calculate the centrifugal force in tons.

The centrifugal force in tons : weight of body in tons : : the centrifugal force in feet $\left(= \frac{v^2}{r} \right)$: gravity in feet; i.e.

Centrifugal force in tons : 17 tons : : $\frac{256}{1110}$: 32.1948.

Hence, centrifugal force equal to . . . . . 0.1217 ton.

4. If a body, weighing 1000 lbs., be constrained to move in a circle, whose radius is 100 feet, by means of a string, capable of sustaining a strain not exceeding 450 lbs., calculate the velocity at the moment the strings breaks. *Ans.* 38.06 ft.

5. If a railway carriage, weighing 7.21 tons, moving at the rate of 30 miles per hour, describe a portion of a circle, whose radius is 460 yards, calculate its centrifugal force in tons. *Ans.* 0.314 tons.

6. Two bodies, weighing respectively 13 lbs. and 16 lbs., move with different velocities on circles whose radii are 11 feet and 15 feet respectively, find the ratio of their velocities when the centrifugal forces exerted upon them are equal. *Ans.* $\frac{4}{5}$ nearly.

7. The time occupied by a body in describing uniformly a complete revolution in a circle, whose radius is 11 feet, is 16 seconds, calculate the centrifugal force which acts upon it. *Ans.* 1.696 ft.

8. If the time occupied in describing 61° of the same circle be 11 minutes, calculate the centrifugal force. *Ans.* 0.000028 ft.

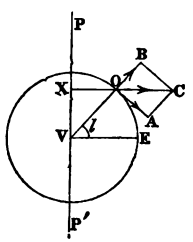
9. If the radius of the circle be 11 yards, and the periodic time 4 minutes, calculate the centrifugal force. *Ans.* 0.0226 ft.

10. If the centrifugal force, in a circle of 100 feet radius, be 146 feet, find the periodic time. *Ans.* 5.2 secs.

11. If the centrifugal force be 131 ounces, and the radius of the circle 100 feet, the periodic time being one hour, calculate the weight of the body. *Ans.* 386.309 tons.

3. **Diurnal rotation of the earth.**—In the diurnal rotation of the earth on its axis we have a

remarkable example of centrifugal force; in this case the periodic time is the same for all parts of its mass, and consequently every particle of the matter composing the earth is repelled from the axis of rotation by a force proportional to its distance from that axis (*vid.* equation 13). It is plain that the centrifugal force arising from the earth's rotation diminishes the attraction which is exerted by the earth on all bodies at its surface. We shall first calculate the effect produced by centrifugal force at the equator, and then describe its effects at other points on the earth's surface. Let PP′ be the axis on which the earth revolves, and VE the equatorial radius equal to R; let G denote the attraction of the earth uninfluenced by rotation, and g as usual denote the *actual* gravity; if f represent the centrifugal force at the equator, we shall have



$$f = G - g;$$

f is known, from equation (13), by substituting for r, the earth's equatorial radius, 20923596 feet, and for T, the number of seconds in a sidereal day, 86164^s; we thus obtain $f = 0.11126$ ft., and since g (the force of gravity at the equator) is about 32.088 ft., we find,

$$f = \frac{g}{288.40};$$

and finally,

$$f = \frac{G}{289.40}. \qquad (14)$$

This may be expressed in whole numbers, by stating that the force of the earth's attraction is 289 times the centrifugal force at the equator.

The effect of the earth's rotation at the other points of its surface may be thus found:—Let O (fig. p. 78) be any point on the surface, OE being its latitude, represented by l; draw OX perpendicular to the axis of rotation PP'; since the periodic time is constant, OX will represent the centrifugal force; produce it until OC is equal to OX; and then resolve the force OC into two components OB and OA, one in the prolongation of the earth's radius, and the other perpendicular to it; the component OB is employed in diminishing gravity, its amount may be thus calculated: OB is equal to OC or OX multiplied by the cosine of BOC, which is equal to the latitude BVE; i. e. $OB = OX \cos l$; but OX is equal to OV (or R) multiplied by the sine of OVX, or the cosine of OVE, or latitude; i.e. $OX = R \cos l$; substituting this value, we obtain

$$OB = R \cos^2 l;$$

but by equation (13) the centrifugal force is equal to OX multiplied by the fraction $\frac{4\pi^2}{T^2}$;

hence we obtain,

$$f = \frac{4\pi^2 R}{T^2} \cos^2 l. \qquad (15)$$

The part of centrifugal force which is employed in diminishing gravity is thus proved to *vary as the square of the cosine of the latitude.* If the latitude

be equal to zero, i. e. if the point be situated on the equator, cos l will be equal to unity, and the centrifugal force will be equal to

$$\frac{4\pi^2 R}{T_2};$$

which has been already shown to be equal to 0.11126 feet; introducing this coefficient, we obtain for the surface of the earth the equation

$$f = 0.11126 \cos^2 l.$$

The component of centrifugal force OA (fig. p. 78), perpendicular to the radius of the earth, is employed in transporting the materials of which the earth's crust is composed towards the equator; this is evident from the direction in which it acts; in consequence of this transporting force, the shape of the earth, even though it were supposed to have been originally spherical, would tend to become somewhat flattened at the poles, and protruding at the equator; the amount of the transporting force is easily calculated: OA is equal to OC or OX multiplied by the sine of BOC or latitude, and OX is equal to $R \cos l$; hence,

$$\mathrm{OA} = R \sin l \cos l,$$

multiplying this equation by $\frac{4\pi^2}{T_2}$, and calling the force we are investigating ϕ; we obtain

$$\phi = \frac{4\pi^2 R}{T_2} \sin l \cos l; \qquad (16)$$

or, for the surface of the earth,

$$\phi = 0.11126 \sin l \cos l.$$

It is easy to prove from this equation that the transporting force will be equal to zero at the equator and at the poles, and will have its greatest value at the latitude of 45°, when it becomes equal to half the centrifugal force at the equator.

EXAMPLES.

1. Calculate the effect of centrifugal force in diminishing gravity in the latitude 60°. By equation (15) the component of centrifugal force employed in diminishing gravity is equal to $\cos^2 60°$ multiplied by 0.11126; $\cos^2 60° = 0.25000$; hence we obtain the component of centrifugal force, = 0.02788 ft.

2. Calculate, for the same latitude, the component of the centrifugal force employed in transporting the materials of the earth's crust to the equator. This is done by means of equation (16). Sin $60° = 0.86602$; $\cos 60° = 0.50000$; multiplying the product of these quantities by 0.11126, we obtain finally the transporting force equal to 0.04817 ft.

3. Calculate the diminution of gravity caused by centrifugal force at the latitude of 23°. *Ans.* 0.09 ft.

4. Calculate the transporting force at the same latitude. *Ans.* 0.04 ft.

5. Calculate both these forces for the latitude of Dublin, viz., 53° 21′.

Force diminishing gravity = 0.0396441 ft.
Transporting force = 0.0532837 ft.

6. Calculate the diminution of gravity at the Sun's equator caused by centrifugal force, the radius of the Sun being 441000 miles, and the time of revolution on his axis being 607 h. 48 m. *Ans.* 0.019200 ft.

7. Calculate the centrifugal force at the equator of Mercury, the radius being 1570 miles, and the time of revolution 24 h. 5 m. *Ans.* 0.04353 ft.

8. Find centrifugal force at the equator of Venus; radius being 3900 miles, and time of revolution 23 h. 21 m. *Ans.* 0.11504 ft.

9. The radius of the planet Mars is 2050 miles, and his periodic time 24 h. 37 m., find centrifugal force at equator. *Ans.* 0.05441 ft.

10. The radius of Jupiter is 43500 miles, and periodic time is 9 h. 56 m., find centrifugal force at equator. *Ans.* 7.09068 ft.

11. The radius of Saturn is 39580 miles, and periodic time is 10 h. 29 m., find centrifugal force at equator. *Ans.* 5.79243 ft.

12. The radius of Uranus is 17250 m., and the periodic time 9 h. 30 m.; find the centrifugal force at his equator. *Ans.* 3.0741 ft.

13. The Moon's distance from the Earth is 59.96435 radii of the Earth; the Earth's diameter is 7925.6 m., and the periodic time of the Moon is 27 d. 7 h. 43 m. 11 s.; find the centrifugal force of the Moon. *Ans.* 0.008889 1 ft.

14. Supposing the Earth's attraction to diminish as the square of the distance, find what gravity becomes at the distance of the Moon, gravity being 32.2 ft. *Ans.* 0.0089551 ft.

	SINE.		TANGENT.	
0°	.00000		.00000	90°
1°	.01745		.01745	89°
2°	.03490		.03492	88°
3°	.05233		.05241	87°
4°	.06975		.06992	86°
5°	.08715		.08749	85°
6°	.10453		.10510	84°
7°	.12187		.12278	83°
8°	.13917		.14054	82°
9°	.15643		.15838	81°
10°	.17365		.17632	80°
11°	.19081		.19438	79°
12°	.20791		.21255	78°
13°	.22495		.23087	77°
14°	.24192		.24933	76°
15°	.25882		.26795	75°
16°	.27563		.28674	74°
17°	.29237		.30573	73°
18°	.30901		.32492	72°
19°	.32557		.34432	71°
20°	.34202		.36397	70°
21°	.35837		.38386	69°
22°	.37460		.40402	68°
23°	.39073		.42447	67°
24°	.40673		.44523	66°
25°	.42262		.46631	65°
26°	.43837		.48773	64°
27°	.45399		.50952	63°
28°	.46947		.53171	62°
29°	.48481		.55431	61°
30°	.50000		.57735	60°
	COSINE.		COTANGENT.	

	SINE.		TANGENT.	
30°	.50000		.57735	60°
31°	.51504		.60086	59°
32°	.52992		.62487	58°
33°	.54464		.64941	57°
34°	.55919		.67451	56°
35°	.57357		.70020	55°
36°	.58778		.72654	54°
37°	.60181		.75355	53°
38°	.61566		.78128	52°
39°	.62932		.80978	51°
40°	.64279		.83910	50°
41°	.65606		.86928	49°
42°	.66913		.90040	48°
43°	.68200		.93251	47°
44°	.69466		.96569	46°
45°	.70710		1.00000	45°
46°	.71934		1.03553	44°
47°	.73135		1.07237	43°
48°	.74314		1.11061	42°
49°	.75471		1.15037	41°
50°	.76604		1.19175	40°
51°	.77714		1.23489	39°
52°	.78801		1.27994	38°
53°	.79863		1.32704	37°
54°	.80901		1.37638	36°
55°	.81915		1.42815	35°
56°	.82904		1.48256	34°
57°	.83867		1.53986	33°
58°	.84805		1.60033	32°
59°	.85716		1.66428	31°
60°	.86602		1.73205	30°
	COSINE.		COTANGENT.	

	SINE.		TANGENT.	
60°	.86602		1.73205	30°
61°	.87462		1.80405	29°
62°	.88295		1.88072	28°
63°	.89100		1.96261	27°
64°	.89879		2.05030	26°
65°	.90631		2.14450	25°
66°	.91354		2.24603	24°
67°	.92050		2.35585	23°
68°	.92718		2.47508	22°
69°	.93358		2.60509	21°
70°	.93969		2.74747	20°
71°	.94552		2.90421	19°
72°	.95105		3.07768	18°
73°	.95630		3.27085	17°
74°	.96126		3.48741	16°
75°	.96592		3.73205	15°
76°	.97029		4.01078	14°
77°	.97437		4.33147	13°
78°	.97815		4.70463	12°
79°	.98162		5.14455	11°
80°	.98481		5.67128	10°
81°	.98769		6.31375	9°
82°	.99027		7.11537	8°
83°	.99254		8.14434	7°
84°	.99452		9.51436	6°
85°	.99619		11.43005	5°
86°	.99756		14.30066	4°
87°	.99863		19.08113	3°
88°	.99939		28.63625	2°
89°	.99985		57.28996	1°
90°	1.00000		Infinite.	0°
	COSINE.		COTANGENT.	

1′	.00029	31′	.00901
2′	.00058	32′	.00931
3′	.00087	33′	.00960
4′	.00116	34′	.00989
5′	.00145	35′	.01018
6′	.00174	36′	.01047
7′	.00203	37′	.01076
8′	.00232	38′	.01105
9′	.00262	39′	.01134
10′	.00291	40′	.01163
11′	.00320	41′	.01192
12′	.00349	42′	.01221
13′	.00378	43′	.01251
14′	.00407	44′	.01280
15′	.00436	45′	.01309
16′	.00465	46′	.01338
17′	.00494	47′	.01367
18′	.00523	48′	.01396
19′	.00552	49′	.01425
20′	.00582	50′	.01454
21′	.00611	51′	.01483
22′	.00640	52′	.01512
23′	.00669	53′	.01541
24′	.00698	54′	.01570
25′	.00727	55′	.01600
26′	.00756	56′	.01629
27′	.00785	57′	.01658
28′	.00814	58′	.01687
29′	.00843	59′	.01716
30′	.00872	60′	.01745

THE END.